A CLASS OF GRAVITY FLOWS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled " A CLASS OF GRAVITY FLOWS", submitted by MAURICE F. ESTABROOKS in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The problem of determining the characteristies of the flow of a fluid with two free streamlines in a gravitational field is investigated. The fluid is assumed to be incompressible and perfect and the flow steady and irrotational. Two cases are considered. The first is such that the whole upper surface is a free streamline. The second is such that the upper surface is partly covered by a solid surface (S_1) extending to infinity upstream. This surface (S_1) and the one (S) on the lower surface are prescribed. The stream is bounded downstream at a finite distance from the point of overflow of the surface (S) in order to avoid a difficulty that occurs in the analysis which occurs when the whole downstream flow is investigated.

The problem is reduced to the determination of a complex-valued function in the unit semi-circle with non-linear boundary conditions.

The solution of the problem is found by the method of successive approximations and then proving that this sequence converges to a unique analytic function satisfying the boundary conditions. The convergence is found to depend upon the Froude number F_r .

The results are applied to a case in which the lower surface (S) consists of a perfectly flat bottom with a vertical wall.

A discussion of the scope of the techniques follows.

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ACKNOWLEDGEMENTS

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CHAPTER I

INTRODUCTION

In this thesis we will investigate a method of solving a class of problems in two dimensional fluid flow under gravity. In order to introduce this topic, it seems advisable to consider the equations with which we will be concerned as well as some of the main concepts of fluid flow. Then a brief discussion on previous work will be presented.

In three dimensions the behavior of a fluid is entirely determined by eleven equations in eleven unknowns. These equations are the equation of continuity, the three equations of motion, the energy equation, the equation of state, and six equations relating the components of the stress tensor to the components of the rate of deformation tensor. The unknown quantities are the density, the three components of the velocity, the six components of the stress tensor, and the pressure or temperature.

The mathematical complexity is usually reduced by assuming that (1) all stresses in the fluid are normal stresses and so reduce to a scalar pressure, in which case the fluid is said to be ideal or perfect and (2) the density is a function of the pressure only, in which case the fluid is said to be barotropic.

The governing equations for two dimensional flow of an ideal barotropic fluid are four.

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \ , \qquad \text{(continuity)}$$

$$\frac{\partial u}{\partial t} + \frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial y} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \ , \qquad \text{(momentum)}$$

$$\frac{\partial v}{\partial t} + \frac{u\partial v}{\partial x} + \frac{v\partial v}{\partial y} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \ , \qquad \text{(momentum)}$$

$$p = p(\rho) \ ,$$

where ρ is the density, u is the x component of the velocity, v is the y component of the velocity, F_x is the x component of body force, F_y is the y component of the body force, p is the pressure, and t is the time.

The fluid motion is said to be steady if all of the flow parameters are independent of time.

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial\rho}{\partial t} = 0 \quad .$$

The fluid motion is said to be irrotational if the curl of the velocity vector is zero; i.e., in two dimensions

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \quad .$$

The fluid is said to be incompressible if the total derivative of the density with respect to the time is zero; i.e., in two dimensions

$$\frac{d\rho}{dt} \equiv \frac{\partial\rho}{\partial t} + \frac{u\partial\rho}{\partial x} + \frac{v\partial\rho}{\partial y} = 0 \quad .$$

When the flow is irrotational it is well known that there exists a function φ called the velocity potential which is such that

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the velocity is equal to the gradient of φ . In two dimensions

$$u = \frac{\partial\varphi}{\partial x}, \quad v = \frac{\partial\varphi}{\partial y} .$$

If in addition to being irrotational, the flow is incompressible, then the equation of continuity reduces to the Laplace equation for the function φ . Hence in two dimensions, the velocity potential is harmonic at all interior points of the fluid. It is possible to construct from $\varphi(x,y)$ its harmonic conjugate $\psi(x,y)$. This function is called the stream function because, as it turns out, ψ is constant along each streamline of the flow. The function $\tilde{F}(z) = \varphi(x,y) + i\psi(x,y)$ of the complex variable $z = x + iy$ is known as the complex potential of the flow. It is regular at all interior points of the fluid domain. As a result of the Cauchy-Riemann equations and the equations relating the velocity components to the velocity potential

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}, \quad u = \frac{\partial\varphi}{\partial x}, \quad v = \frac{\partial\varphi}{\partial y},$$

we deduce that

$$w \equiv u - iv = \frac{d\tilde{F}}{dz},$$

where w is known as the complex velocity. Hence once $\tilde{F}(z)$ is known, then the velocity distribution can be found at once.

Usually the body force $\vec{F}$ is assumed to be conservative in which case it is well known that there exists a scalar function U such that $\tilde{F}$ is the gradient of $(-U)$.

In this work we will assume that (1) the fluid is perfect,

barotropic, and incompressible, (2) the flow is steady and irrotational, (3) the body force is conservative, (4) the density ρ is constant and the value of this constant is 1 , and (5) the flow can be restricted to a two dimensional flow. Under these assumptions the flow is governed by the following equations,

$$\frac{\partial\varphi}{\partial t} = 0 \quad ,$$

$$(u,v) = \left(\frac{\partial\varphi}{\partial x} , \frac{\partial\varphi}{\partial y}\right) \quad ,$$

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \quad ,$$

$$U + \frac{p}{\rho} + \frac{1}{2}(u^2 + v^2) = \text{constant} \ .$$

The last equation is known as the Bernoulli equation. Using it the pressure can be computed once the velocity distribution has been found.

On each boundary of the flow domain it is necessary that some condition be satisfied. Perhaps the pressure distribution or the normal velocity distribution (e.g. $\partial\varphi/\partial n = 0$ along a fixed solid boundary) is known. In the case of a streamline along which the pressure is constant, the Bernoulli equation supplies the boundary condition. Such a streamline is called a free streamline and the boundary is called a free boundary.

The following is a list of symbols, with their meanings, which will be used extensively throughout the remainder of this work.

ρ is the density,

p is the pressure,

u is the x component of the velocity,

v is the y component of the velocity,

$w = u - iv$ is the complex velocity,

$V = (u^2 + v^2)^{1/2}$ is the magnitude of the velocity,

$\vec{V}$ is the velocity vector,

$\vec{F}$ is the external body force,

U is the gravitational potential,

g is the acceleration due to gravity,

φ is the velocity potential,

ψ is the stream function,

$\tilde{F}$ is the complex potential,

$\omega = f + ig$ the Levy-Civita function,

$\omega_n = f_n + ig_n$ the sequence of successive approximations of ω,

$\Omega_n = F_n + iG_n \equiv \omega_n - \omega_1$ for $n > 1$.

Other symbols which may occur will be defined as they arise.

CHAPTER II

PREVIOUS WORK

The study of the motion under gravity of a fluid with a free upper surface has drawn considerable attention because of its practical significance. Although numerical results are available, the determination of a general mathematical theory incorporating all of the parameters of a real flow is difficult.

In special cases, for example the flow of a fluid over a sharp angle, the flow of ideal fluids leads to theoretical results which agree satisfactorily with experimental results.

H. Villat [1] has constructed a basis for the study of the motion under gravity of a fluid with a free surface while H. Poncin [2] has done a great deal of work on this problem.

For a review of the known theorems to date see Hyers [5] and Birkhoff [9].

In his work [2] Poncin considers the steady irrotational flow of an ideal incompressible fluid under gravity when the flow is supported by a surface (S), the bed of the stream on which is specified the angle the velocity vector makes with the horizontal. The flow also has a single free streamline which forms the upper surface of the fluid domain.

We shall consider Poncin's paper.

In part A the case where the motion is periodic is considered. This does not concern us directly.

In part B the case where the flow extends to infinity upstream and downstream is considered. The flow is horizontal and the velocity is constant at the extremities of the fluid domain.

In part C is considered the generalizations of the problem in part B, in which the upper surface is partly covered by a solid surface (S_1) and when on the lower surface (S), the fluid leaves the bed or the surface (S) temporarily, and returns to it after a short distance.

The description of periodic waves does not concern us directly in this discussion, but it is interesting to note that the problem reduces to either solving an integral equation or solving a mixed boundary value problem where the boundary values are non-linear. No attempt is made to solve the integral equation. The boundary value problem is solved by the method of successive approximations.

In part B of the work, the fluid domain extends to infinity upstream and downstream where the flow is horizontal and the velocity assumes constant values $V(A)$ and $V(B)$ respectively. The upper surface is a free streamline and on (S), the lower surface, the slope of the velocity is prescribed. The region occupied by the fluid in the complex potential plane $\tilde{f}$ is mapped conformally into the interior of the upper half of the unit circle $|z| = 1$ of the complex z plane by mapping

$$\tilde{f} = -\frac{iQ}{2} + \frac{2Q}{\pi} \operatorname{Log} \left\{\frac{1+z}{1-z}\right\} . \qquad (2.1)$$

With the complex velocity given by expression

$$w(z) = V(A)\, e^{-i\omega(z)} \tag{2.2}$$

where

$$\omega(z) = f(x,y) + ig(x,y) \quad ,$$

the problem reduces to the determination of an analytic function $\omega(z)$ which is regular in the interior of the upper unit semi-circle and satisfies non-linear boundary conditions on its boundary $|z| = 1$. The boundary value problem is solved by the method of successive approximations. (See the details below.) The arguments $(x,0)$ will refer to the diameter of the upper unit semi-circle for $-1 \leqq x \leqq +1$ while the arguments $(1,s)$ will refer to the circumference of this unit semi-circle when $r = 1$ and $0 \leqq s \leqq \pi$ for s the arc length. On the boundary of this upper unit semi-circle the function $\omega_n(z)$ under the above notation must satisfy,

$$\omega_n(-1) = 0 \ ,$$

$$f_n(x,0) = \theta(x) \quad \text{the diameter,} \tag{2.3}$$

$$\frac{df_n}{dr}(1,s) = \frac{\mu \sin f_{n-1}(1,s)\, e^{-3g_{n-1}(1,s)}}{\sin s} \quad \text{on the circumference} \ ,$$

where μ is a parameter equal to $gQ/(\pi V^3(A))$ and $\theta(x)$ is prescribed on the diameter of the unit semi-circle. On the boundary of this semi-circle the solutions are

$$f_1(1,s) = \frac{2 \sin s}{\pi} \int_{-1}^{+1} \frac{\theta(u)}{u^2-2u \cos s+1} \, du \quad ,$$

$$g_1(1,s) = 0$$

$$f_1(x,0) = \theta(x) \quad ,$$

$$g_1(x,0) = \frac{1}{\pi} \int_{-1}^{+1} \frac{\theta(u)(x^2-1)}{(x-u)(1-ux)} \, du \quad ,$$

$$f_n(1,s) = f_1(1,s) + \frac{\mu}{\pi} \int_o^{\pi} N_n(u) \operatorname{Log} \left|\frac{\sin (u+s)/2}{\sin (u-s)/2}\right| du$$

$$g_n(1,s) = \mu \int_o^s N_n(u) du \tag{2.4}$$

$$f_n(x,0) = \theta(x)$$

$$g_n(x,o) = - \frac{2\mu}{\pi} \int_o^{\pi} N_n(u) \operatorname{Arctan} \left\{\frac{1+x}{1-x} \operatorname{Tan} \frac{u}{2}\right\} du$$

where

$$N_n(u) = \frac{\sin f_{n-1}(1,u)}{\sin u} \, e^{-3g_{n-1}(1,u)} \quad .$$

In order that $\omega_1(z)$ be well behaved, it is necessary that $\theta(x)$ be of the form

$$\theta(x) = \theta_o(x)(1-x^2) \tag{2.5}$$

where $\theta_o(x)$ is bounded in $-1 \leq x \leq 1$. This means that the flow must be horizontal at infinity upstream and downstream. The sequence $\{\omega_n(z)-\omega_1(z)\}$ is shown to be uniformly bounded on the boundary of the

semi-circle and uniformly convergent to a unique limit function which is regular in the interior of the domain D and satisfies the boundary conditions of the problem providing μ satisfies the following relation

$$\mu < \frac{2}{\pi} \exp\left(-m+3G_m\right), \qquad 3G_m = 1-e^{-m},$$

$$m = 2 \operatorname{Arg\,Sh} \sqrt{3J_1/(2\pi)}, \qquad J_1 = \int_0^{\pi} \left|\frac{f_1(1,s)}{\sin s}\right| ds \quad . \tag{2.6}$$

The pressure distribution is analysed throughout the fluid domain. In order that the solution be acceptable in a hydrodynamical sense it is essential that the pressure be non-negative at all points of the fluid. For irrotational flow of an incompressible ideal fluid under gravity with constant density, it is evident from the Bernoulli equation that $\Delta p \leqq 0$. This means that the pressure is a superharmonic function in the interior of the fluid domain. (See Courant and Hilbert, "Methods of Mathematical Physics", pages 307 and 326.) By the Minimum Principle for superharmonic functions, the pressure must assume its minimum on the boundary of this domain. It is necessary to show that the pressure is positive on the boundary of the flow, in which case it will be positive at all interior points.

For the limiting case and for this case only the pressure is constant along the free streamline (L) and this constant is non-negative if the pressure infinitely far upstream on this streamline is non-negative. For any approximation $\omega_n(z)$ the difference between the pressure at any point on (L) and the point infinitely far upstream can be made as small as possible providing n is made large.

On (S) the study of the pressure reduces to the study of the behavior of $g_1(x,0)$. If $g_1(x,0)$ is bounded then the solution is acceptable in the hydrodynamical sense. This is particularly evident if $\theta(x)$ has a discontinuity of the first kind at some point of (S) . In this case $g_1(x,0)$ becomes infinite with the sign corresponding to $\theta(x-0)-\theta(x+0)$. Hence if the lower surface (S) presents angles pointing into the fluid region, the solution is not acceptable because the pressure becomes negative. The case examined in the work of a surface (S) which is horizontal upstream and downstream but experiences a vertical drop at some point of the flow serves to illustrate this. At the point which experiences this vertical drop, the pressure becomes negative. Further analysis shows that exterior to a neighbourhood of this point whose radius is of the order of 10^{-4} cm , the pressure is positive. Hence one can find a streamline passing immediately outside this neighbourhood at all points of which the pressure is positive. In actual fact the radius of the above neighbourhood is negligible in comparison to the precision of other measurements occurring in the problem. So, for this reason, this streamline may be made to represent the lower boundary of the flow. The solution then obtained is such that the pressure is positive along this boundary and hence inside the fluid.

As we shall see, if $\theta(x)$ is continuous and differentiable, then $g_1(x,0)$ is well behaved and the solution is valid in the hydrodynamical sense.

In part C a generalization is made. The fluid is interior to two horizontal bands which are a finite distance apart. Along these contours the fluid moves on a solid surface (S), a fluid surface (G)

of the same nature but at rest, or else it is in contact with the atmosphere (L) where the pressure is constant. The previous analysis can easily be adapted to account for cases arising in this problem.

In chapter one of part C the case of a canal partially covered upstream by a surface (S_1) is considered.

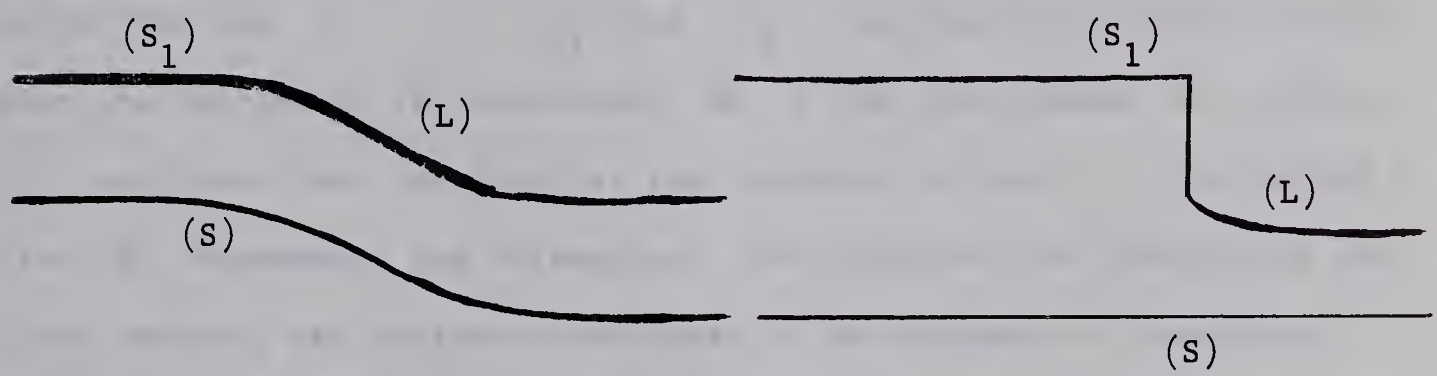


Figure 1

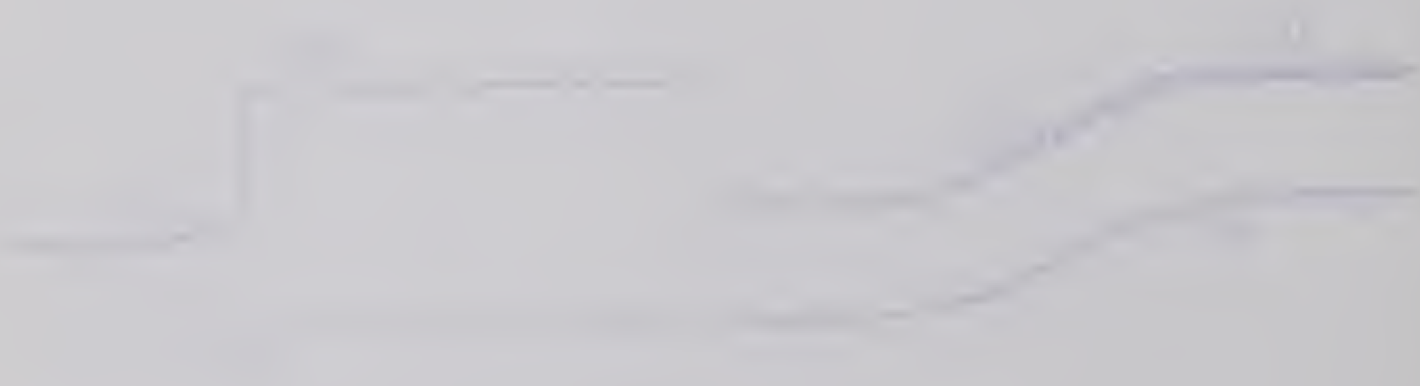
He deals specifically with the two cases when θ as described on this upper surface is monotonically decreasing and when θ is such that the surface (S_1) is horizontal and experiences a vertical drop as indicated in the diagram above (Figure 1).

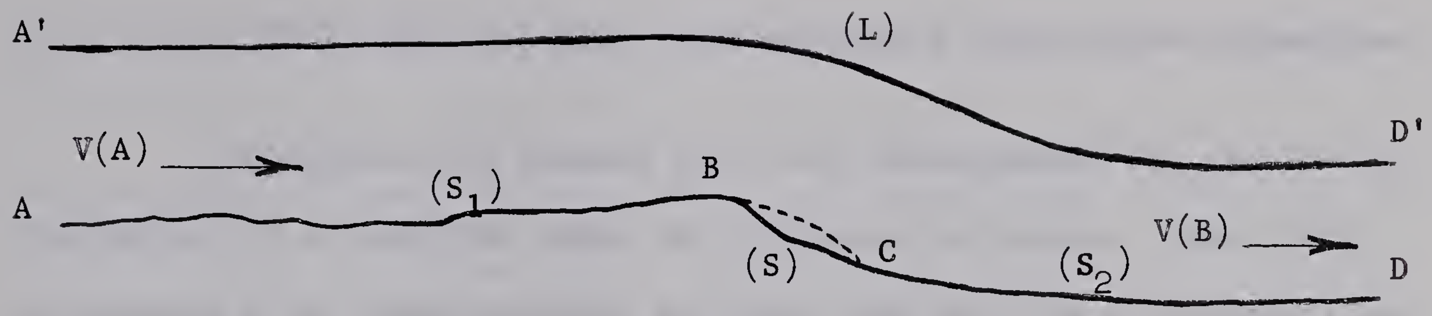
The normal derivative

$$\mu\, e^{-3g(1,s)} \frac{\sin f(1,s)}{\sin s}$$

is somewhat more complicated than before but poses no additional difficulties.

Chapter three of part C deals with the case where the contours of the flow consist of four arcs. Refer to the figure below.





(L) is the upper free streamline. (S_1) and (S_2) form part of the solid flow bed (S) . On (S_1) and (S_2) the angle the velocity makes with the horizontal is prescribed. At B the flow leaves the surface (S) and flows over the fluid at rest between B and C . The dotted line BC represents the streamline. The solutions are obtained as outlined earlier, the uniform boundedness of the sequence of successive approximations on the boundary of the fluid as well as its convergence follow in a similar manner as above. Bounds on the parameter μ are obtained and the pressure distribution is obtained as before with similar conclusions.

Using the results of Poncin's work Dumitrescu, Ionescu, and Craciun [10] adopt techniques in the study of inverse limits [14], [15] and [16] to study the problem of determining the characteristics of the flow bounded by a solid surface below and a free streamline above both extending to infinity upstream and downstream.

In general certain restrictions must be placed on θ to obtain satisfactory results. This has been evident in the papers discussed.

Although the motion of a fluid bounded by two free streamlines is somewhat more practical than that bounded by one not much work has been done on it in comparison to the former. The papers [2], [14], [18],

[19], [20], [21], [22] and [23] consider only a single free streamline.

Dumitrescu and Ionescu [13] have investigated the problem of the motion of an overflow under the influence of gravity. The fluid is assumed to be incompressible and ideal and the flow irrotational and steady. In order to avoid a theoretical difficulty what would occur if the whole infinite flow were considered, the authors bound the flow at a finite distance downstream of the point of overflow. However the flow extends infinitely far upstream where it is horizontal with magnitude $V(A)$. The flow region (Figure 2) is bounded by two free streamlines (L_1) and (L) , an equipotential surface CC_1 and a fluid bed (S) on which θ is specified.

The region $-\infty < \varphi < +\infty$, $0 \leqq \psi \leqq Q$ of the complex potential plane $\tilde{f} = \varphi + i\psi$ representing the fluid domain is mapped into the upper unit semi-circle in the z plane and the boundary conditions found. The mapping and normal derivative of the real part of $\omega(z)$ on the circumference of the semi-circle are

$$\tilde{f} = \frac{Q}{\pi} \operatorname{Log} \frac{(1+z)^2}{2(1+z^2)}$$

$$\frac{df}{dr}(1,s) = -\mu \frac{\operatorname{Tan} s/2}{\operatorname{Cos} s} e^{-3g(1,s)} \sin f(1,s) \quad .$$

The other boundary conditions correspond to one and two of (3.3). The boundary condition above is more complex than any other one we have seen to this stage. Once the function $\omega(z)$ has been found to be regular in the upper unit semi-circle and to satisfy the boundary conditions then the streamlines can also be found.

Utilizing the method of successive approximations the authors solve the boundary value problem in the same way as Poncin does. The solution is expressed as the limit of a sequence of holomorphic functions $\omega_n(z)$. However, in examining the uniform boundedness and convergence of this sequence, the even approximations are such that a singularity always occurs at $\pi/2 = s$ which lies on the circumference of the upper unit semi-circle D . The approximations are otherwise very well behaved. Due to this fact one theoretically can deduce nothing about convergence of the sequence to a solution of the problem. But the theoretical results agree satisfactorily with experimental results provided one restricts himself to a vicinity of the overflow.

The problem is approached in a slightly different manner in a paper by Dumitresu, Ionescu and Craciun [8]. Only the semi-infinite strip $0 \leq \psi \leq Q$, $-\infty < \varphi \leqq 0$ of the complex potential plane is mapped onto the new region. This region is the upper half of the z' plane. The surface (S) of the flow corresponds to the interval $[1,\alpha]$ where α is determined by the mapping. The boundary conditions are found and the method of successive approximations is used as above to solve the problem. The results will appear below later.

In order to solve the problem for approximation other than the first, the authors map the upper half plane z' onto the upper unit semi-circle of the z plane. The normal derivative of the function $F_n(x',y') = \text{Re}\ \{\omega_n(z')-\omega_1(z')\}$ is mapped into the upper unit semi-circle of the z plane and corresponds to the outward normal derivative on the circumference of this semi-circle. The solution is found for this semi-circle and the results are mapped back to the z' plane.

I have found the problem can be simplified if one deals only with the semi-circle and dispenses completely with the upper half z' plane. The results which I received for the upper unit semi-circle were then mapped onto the upper half plane. They are not identical with those obtained in reference [8]. With the functions $N(\eta, z')$ and $\omega_1(z')$ defined by (5.12) and (5.3) the results that I obtained for the approximation to order n were $\omega_n(z')$ equal to

$$\omega_1(z') + \frac{\mu}{\pi} \int_D \frac{\operatorname{Sin} f_{n-1}(\eta,0)\, e^{-3g_{n-1}(\eta,0)}}{-\sqrt{\eta}\,(\eta-1)} \operatorname{Log} \frac{N(\eta,z')}{N(\eta,1)}\, d\eta$$

while those of the authors of refernce [8] are

$$\omega_1(z') + \frac{\mu}{\pi} \int_D \frac{\operatorname{Sin} f_{n-1}(\eta,0)\, e^{-3g_{n-1}(\eta,0)}\,(\alpha-1)}{\sqrt{\eta}\,(1-\eta)(2\eta-\alpha-1)\sqrt{(\eta-1)(\eta-\alpha)}} \operatorname{Log} \frac{N(\eta,z')}{N(\eta,1)}\, d\eta \quad .$$

The integrals in the latter case do not appear to be convergent while those in the former case are convergent as we shall see below.

In [8] no proof of the convergence of the sequence of successive approximations is attempted although it is stated explicitly that the convergence does depend upon the parameter μ .

In this paper the whole problem will be re-examined and all of the points discussed immediately above will be clarified. Although an argument similar to Poncin's could be used to prove the uniform boundedness and uniform convergence of the sequence of successive approximations we shall use a different approach because the kernels of the functions obtained here are more complicated and require more

work than those used by Poncin.

Some numerical computations are performed by hand calculators and two examples are partially examined. The first is a vertical weir and the second is a type of horizontal jet emitted from an infinite rectangular box. The purpose of these computations is to examine the class of flows to which this theory is applicable. Also some results are compared to those of paper [8] in order to determine the difference in the behavior of the results obtained in each case. Finally we wish to investigate the computational difficulties if any and determine the feasibility of using an electronic computor to do most of the computation.

CHAPTER III

FORMULATION AND OUTLINE OF SOLUTION OF PROBLEM

Mathematical Hypothesis

Consider steady irrotational flow in two dimensions of an ideal incompressible fluid under gravity. With reference to Figure 2 the stream is bounded by the streamlines (L) and (L_1) and the equipotential line CC' as well as the solid surfaces (S_1) and (S) which are of course also streamlines. The upper surface (S_1) is horizontal here and fixed but could be arbitrary apart from the fact that it becomes horizontal at infinity. The lower surface is such that it tends toward the horizontal direction asymptotically in the upstream direction.

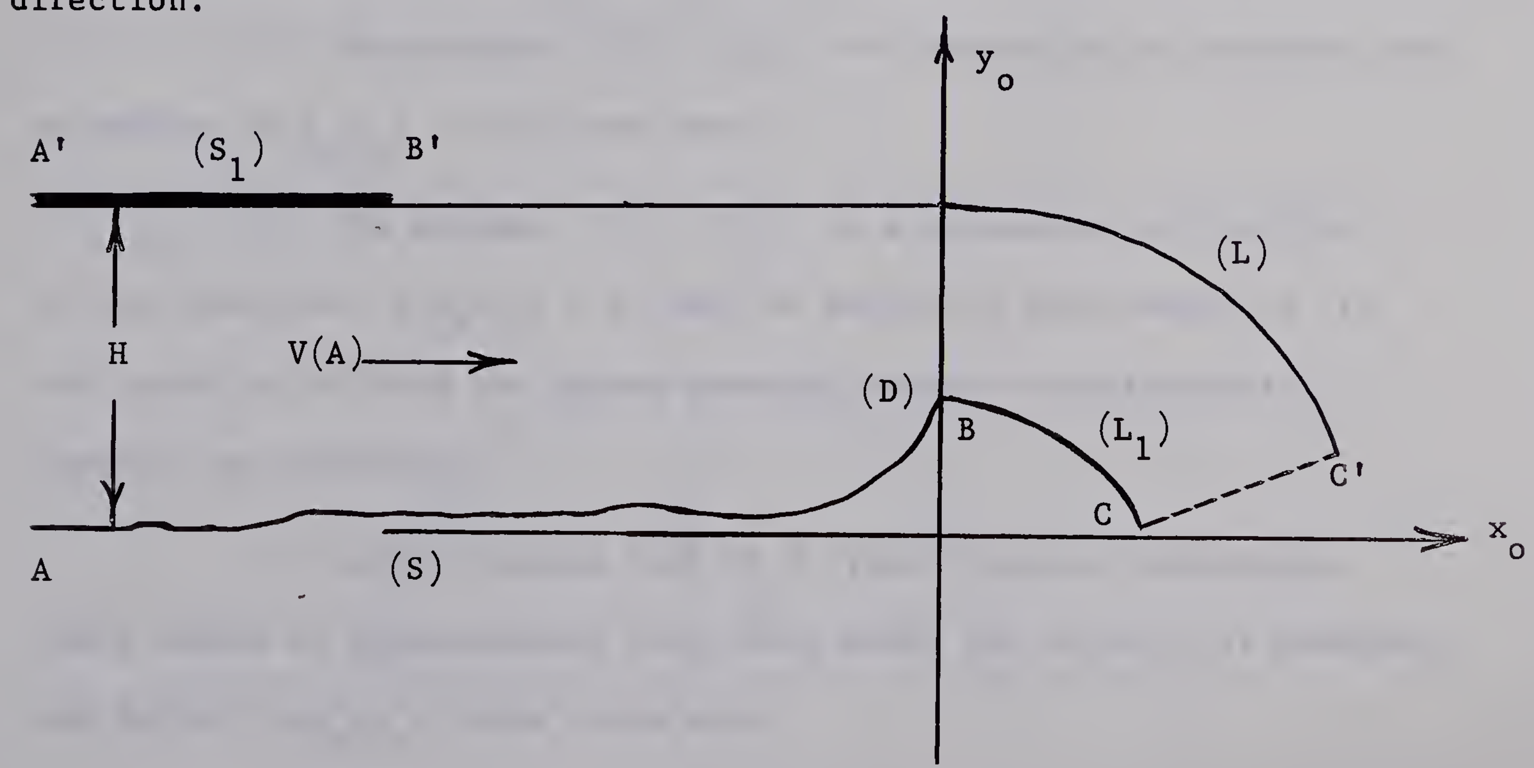


Figure 2

Let $V(A)$ denote the velocity at infinity upstream and H the fluid height at this point. We will assume the density is 1 and that the fluid remains in contact with the surface (S). If $\ell = \ell(x_o, y_o)$ is the equation of (S), then dy_o/dx_o = slope $\vec{V} \equiv \theta$ on (S).

After the fluid leaves point B, it falls freely in a gravitational field forming the free streamline (L_1). A similar description applies to the case of B', and it forms the free streamline (L). The stream is bounded at a finite distance downstream from B by the equipotential CC' in order to avoid a difficulty that would occur in the theory if the whole downstream flow were considered.

Boundary Conditions

Under the above assumptions, the problem is confined to the determination of a complex valued function $\tilde{f}(x_o, y_o)$ defined in the fluid domain of figure 2 which satisfies the following boundary conditions.

a) The surface $(S) + (L_1)$ is a streamline of the flow and we define $\psi(x_o, y_o)$ to be zero there.

b) The surface $(L) + (S_1)$ is a streamline of the flow so the condition $\psi(x_o, y_o) = Q$ must be satisfied there where Q is the quantity of fluid per second passing through an equipotential surface one unit wide.

c) We will assume that at a finite distance downstream there exists an equipotential line along which the velocity is constant, and define $\varphi(x_o, y_o)$ there to be zero.

d) The free surfaces (L) and (L_1) must be determined in such a way that on them the Bernoulli equation is satisfied. i.e.

$$V^2 + 2gy_0 = \text{constant} \quad . \tag{3.1}$$

The region occupied by the flow is represented in the complex potential plane $\tilde{f} = \varphi + i\psi$ by the lettered region ABCC'B'A' in Figure 3 where the points in Figures 2 and 3 are made to correspond by identical lettering.

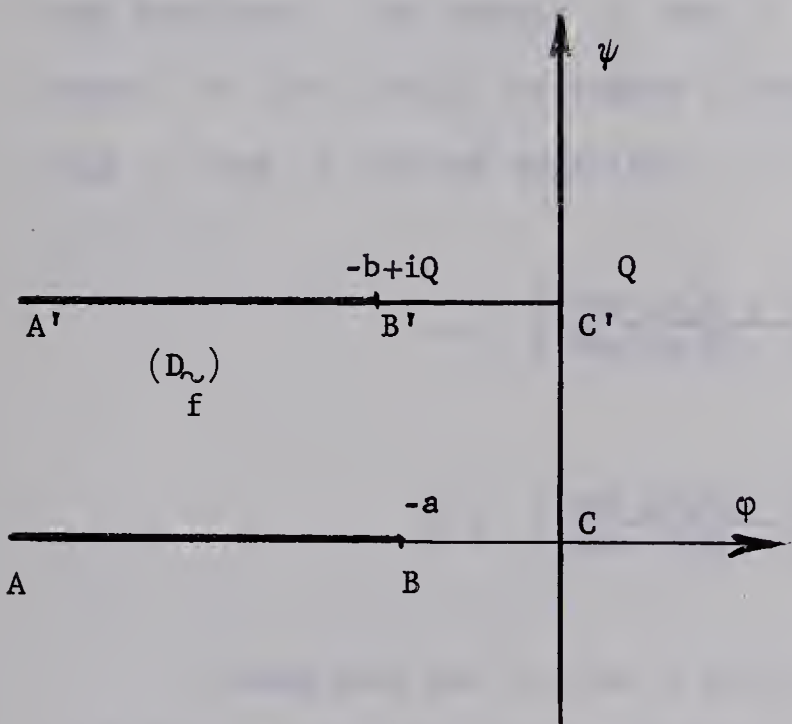


Figure 3

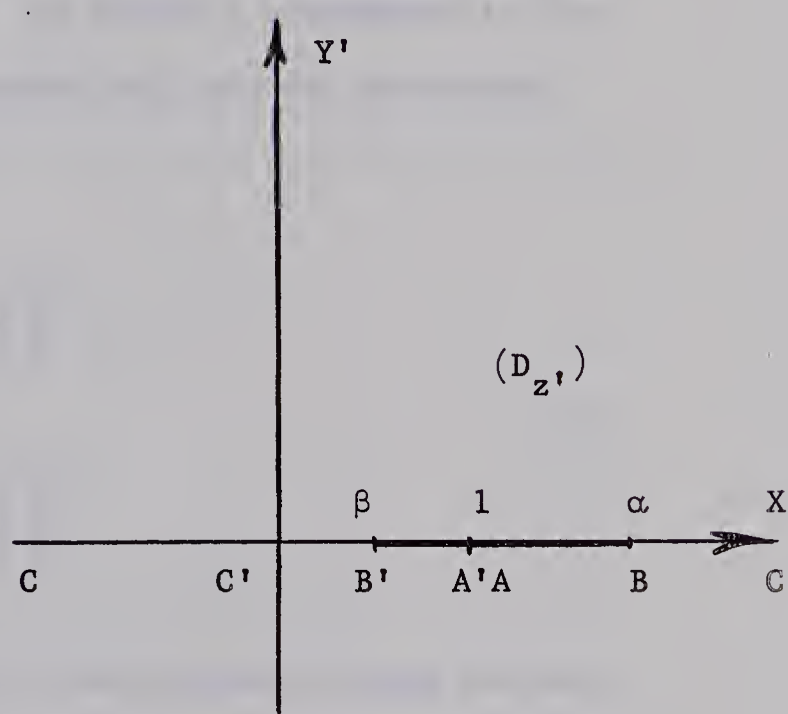


Figure 4

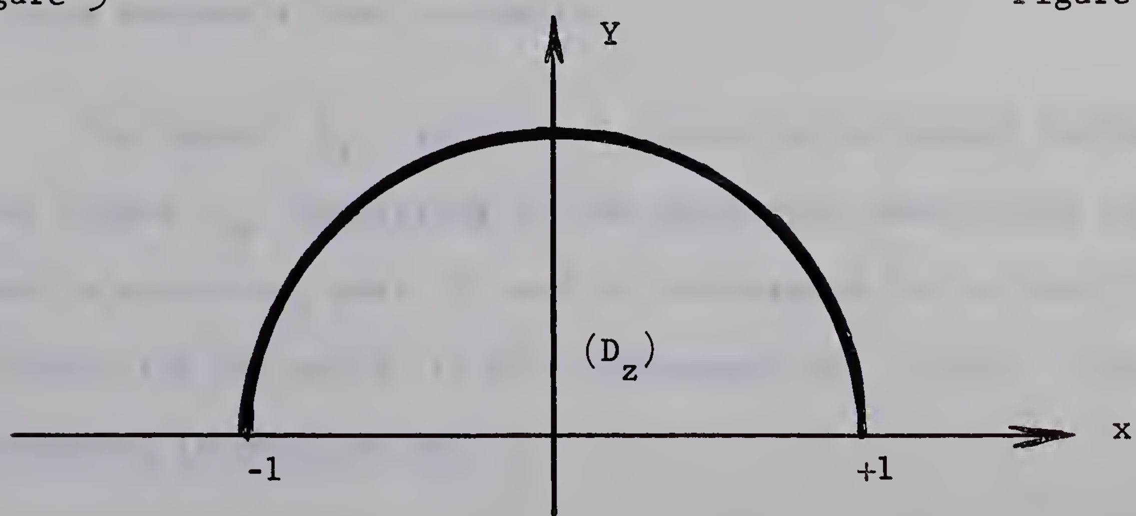


Figure 5

The transformation

$$z' = \left\{ \frac{1 + \exp \pi\tilde{f}/Q}{1 - \exp \pi\tilde{f}/Q} \right\}^2 \quad , \tag{3.2}$$

maps the region $D_{\tilde{f}}$ of the complex potential $\tilde{f}$ plane conformally into the upper half of the complex z' plane $(D_{z'})$. See Figures 3 and 4.

The quantities a and b can be approximated or they may be made arbitrary. The points α and β of Figure 4 correspond to the points $-a$ and $-b+iQ$ of Figure 3 respectively and are determined from a and b by the formulae:

$$\begin{aligned} \alpha &= \left\{ \frac{\exp \pi a/Q + 1}{\exp \pi a/Q - 1} \right\}^2 \quad , \\ \beta &= \left\{ \frac{\exp \pi b/Q - 1}{\exp \pi b/Q + 1} \right\}^2 \quad . \end{aligned} \tag{3.3}$$

Note that as $b \to +\infty$, $\beta \to +1$ and the whole upper surface of the flow becomes a free streamline.

The region $D_{z'}$ in the z' plane now is mapped conformally onto the region D_z consisting of the upper unit semi-circle in the z plane in such a way that β and α correspond to -1 and $+1$ respectively and the strip $[\beta,\alpha]$ corresponds to $[-1,+1]$. The transformation is found to be:

$$z = \left(\frac{\alpha-\beta}{2} + i\sqrt{(z'-\alpha)(z'-\beta)} \right)\left(z' - \frac{\alpha+\beta}{2} \right)^{-1} \tag{3.4}$$

where the square root is chosen to be real when z' is real and greater than α. The inverse $\tilde{f} = \tilde{f}(z')$ of (3.2) and the inverse $z' = z'(z)$ of (3.4) can be computed, and $\tilde{f}(z)$ is found to be

$$\tilde{f}(z) = \frac{Q}{\pi} \operatorname{Log} \left\{ \frac{\left(\frac{\alpha+\beta}{2} + \frac{(\alpha-\beta)z}{1+z^2} \right)^{\frac{1}{2}} - 1}{\left(\frac{\alpha+\beta}{2} + \frac{(\alpha-\beta)z}{z^2+1} \right)^{\frac{1}{2}} + 1} \right\} , \qquad (3.5)$$

where we choose the branch $-\frac{\pi}{2} < \operatorname{Arg} \sqrt{z'} < \frac{3\pi}{2}$ and the branch of the logarithm to be such that $-Q < \operatorname{Im} \tilde{f} < +Q$.

It is convenient to introduce the Levy-Civita function $\omega(z)$ defined by

$$w(z) = V(A) \exp(-i\omega(z)) , \qquad (3.6)$$

$$\omega(z) = f(x,y) + ig(x,y) , \qquad (3.7)$$

where $w(z)$ is the complex velocity. We must determine $\omega(z)$ from the boundary condition on the boundary of the unit semi-circle in the z plane. Denoting the arc length on the circumference of the semi-circle in the z plane by s we obtain:

$$\begin{aligned} V(1,s) &= V(A) \exp g(1,s) , \\ \theta(1,s) &= f(1,s) , \end{aligned} \qquad (3.8)$$

which when differentiated yields:

$$dV(1,s) = V(A)g'(1,s)\exp g(1,s)ds \quad ,$$

$$\text{or} \qquad g'(1,s) = \frac{dV}{ds}\;\frac{\exp\,(-g(1,s))}{V(A)} \quad . \tag{3.9}$$

From the velocity relationship

$$\frac{d\tilde{f}}{dz_o} = V\exp\,(-i\theta) \quad , \tag{3.10}$$

where $z_o = x_o + iy_o$ denotes the original physical plane of the flow we obtain

$$\frac{dz_o}{d\tilde{f}} = \frac{dx_o}{d\varphi} + i\,\frac{dy_o}{d\varphi} = \frac{1}{V(\varphi)}\exp i\theta(\varphi) \ , \tag{3.11}$$

along a streamline ψ = constant.

Equating the imaginary parts we obtain

$$V(\varphi)dy_o = \sin\,\theta(\varphi)d\varphi \quad . \tag{3.12}$$

Now differentiate the Bernoulli equation (3.1)

$$VdV + gdy_o = 0 \ , \tag{3.13}$$

and into this substitute (3.12) to obtain

$$V^2(\varphi)dV + g\,\sin\,\theta(\varphi)d\varphi = 0 \quad .$$

Substitute this into (3.9) for $g'(1,s)$ to obtain

$$g'(1,s) = -\frac{d\varphi}{ds}\,\frac{g}{V(A)}\;\frac{\sin\,\theta(\varphi)}{V^2(\varphi)}\exp\,(-g(1,s)) \ . \tag{3.14}$$

Now compute $\frac{d\varphi}{ds}$ from equation (3.5), let $\cos\gamma = -(\alpha-\beta)/(\alpha+\beta)$.

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$$\frac{d\varphi}{ds} = \mathrm{Re}\ \left\{\frac{d\tilde{f}}{ds}\right\} = \mathrm{Re}\ \left\{\frac{d\tilde{f}}{dz} \times \frac{dz}{ds}\right\} \quad ,$$

$$= \mathrm{Re} \left\{ \frac{Q(\alpha-\beta)}{\pi} \times \frac{1-z^2}{(1+z^2)^2} \times \frac{iz}{\left(\frac{\alpha+\beta}{2} + \frac{(\alpha-\beta)z}{1+z^2} - 1\right)\sqrt{\frac{\alpha+\beta}{2} + \frac{(\alpha-\beta)z}{1+z^2}}} \right\} ,$$

$$= \mathrm{Re} \left\{ \frac{Q}{\pi} \frac{(\alpha-\beta)\ \mathrm{Tan}\ s}{\{(\alpha+\beta-2)\ \mathrm{Cos}\ s + \alpha-\beta\} \sqrt{\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2\ \mathrm{Cos}\ s}}} \right\}$$

$$= -\frac{Q}{\pi} R_1(S) \ , \quad \text{where} \tag{3.15}$$

$$R_1(S) = \begin{cases} 0 & \text{if} \quad \gamma > s > \pi/2 , \\ \dfrac{-\ \mathrm{Tan}\ s}{\left\{ \dfrac{(\alpha+\beta-2)}{(\alpha-\beta)}\ \mathrm{Cos}\ s + 1 \right\} \sqrt{\dfrac{\alpha+\beta}{2} + \dfrac{\alpha-\beta}{2\ \mathrm{Cos}\ s}}} & \text{if} \quad \pi > s > \gamma , \\ & \quad \dfrac{\pi}{2} > s > 0 . \end{cases}$$

Substitute equation (3.8) for $\theta(\varphi)$ and $V(\varphi)$ as well as (3.15) into equations (3.14) to obtain:

$$g'(1,s) = +\mu R_1(s)\ \sin f(1,s)\ \exp\ (-3g(1,s)) \quad , \tag{3.16}$$

where

$$\mu = \frac{gQ}{\pi V^3(A)} = \frac{gH}{\pi V^2(A)} = \left(\frac{1}{\pi F_r} \right) \quad , \tag{3.17}$$

F_r being the Froude Number. Here $R_1(s)$ and hence $g'(1,s)$ is zero where s corresponds to the equipotential line CC' which is shown in Figure 2 so $\gamma \geq s \geq \pi/2$.

Now infinitely far upstream the velocity is $V(A)$ so from expression (3.6) $\omega(z)$ at this point must be zero. In the z' plane

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it corresponds to $z' = +1$ so in the z plane it corresponds to $z = \delta$ where

$$\delta = \frac{\alpha-\beta-2\sqrt{(1-\beta)(\alpha-1)}}{2-\alpha-\beta} \quad , \tag{3.18}$$

for α and β are now known quantities.

Note as the upper surface (S_1) recedes to infinity, then $\beta = 1$ and $\delta \to -1$ so in this case the problem reduces to the flow with an upper free streamline extending to infinity upstream.

The problem can now be formulated as follows. The function $\omega(z)$ is to be determined so as to be regular in the upper unit semi-circle of the z plane and to fulfill the following conditions on the boundary of this semi-circle:

$$\begin{aligned}
&\omega(\delta) = 0 \qquad \delta = \frac{\alpha-\beta-2\sqrt{(1-\beta)(\alpha-1)}}{2-\alpha-\beta} \\
&f(x,0) = \theta(x) \quad \text{on the diameter} \quad -1 \le x \le +1 \\
&g'(1,s) = +\mu R_1(s) \sin f(1,s) \exp\left(-3g(1,s)\right) \quad \text{on the circumference} \\
&\qquad\qquad 0 \le s \le \pi
\end{aligned} \tag{3.19}$$

where

$$R_1(s) = \begin{cases} 0 & \text{for} \quad \gamma > s > \frac{\pi}{2}, \\[2ex] \dfrac{-\operatorname{Tan} s}{\left(\frac{\alpha+\beta-2}{\alpha-\beta}\cos s+1\right)\sqrt{\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2\cos s}}} & \text{for} \quad \pi > s > \gamma,\ \frac{\pi}{2} > s > 0. \end{cases}$$

The first condition means that at infinity upstream the velocity is constant and is $V(A)$. The second condition means that the slope

of the velocity is given on the two surfaces (S) and (S_1) and is represented on the diameter of the upper unit semi-circle by $\theta(x)$. We can assume that (S_1) is perfectly flat in which case $\theta(x) = 0$ for $-1 \leq x \leq \delta < 0$. Indeed we do this in order to make the computation easier and to lessen the difficulties in proving uniform boundedness and uniform convergence of our sequence of successive approximations which will be obvious later. For the general case, i.e. for $\theta(x)$ prescribed arbitrarly on the surface (S_1) it is easily seen in the following discussion that this reduces to the case when $\beta = 1$ and $\delta = -1$. The third condition is just the Bernoulli equation for the z plane.

Also, once $\omega(z)$ is found then the velocity distribution $\vec{V}(z)$ can be easily found from equation (3.6) and the boundaries of the flow will be determined by equation (3.21) which we now deduce.

From equations (3.6) and (3.10) we obtain

$$dz_o = \frac{1}{V(A)} \exp i\omega(z) \frac{d\tilde{f}}{dz} dz ,$$

where $\frac{d\tilde{f}}{dz}$ is obtained from (3.5). Hence

$$dz_o = \frac{Q}{\pi V(A)} \times \frac{1-z^2}{(1+z^2)^2} \times \frac{\exp(i\omega(z))}{\left(\frac{\alpha+\beta-2}{\alpha-\beta} + \frac{z}{1+z^2}\right)} \times \frac{dz}{\sqrt{\frac{\alpha+\beta}{2} + \frac{(\alpha-\beta)z}{1+z^2}}} . \quad (3.20)$$

Now to obtain the streamlines of the flow and the boundaries of the flow we integrate equation (3.20). For example, the boundaries of the flow are found by integrating along the boundaries of the unit semi-circle.

$$z_o = \frac{Q}{\pi V(A)} \int_{x_1}^{x} \frac{1-x^2}{(1+x^2)^2} \frac{\exp\,(i\omega(x))}{\left(\frac{\alpha+\beta-2}{\alpha-\beta} + \frac{x}{1+x^2}\right)} \frac{dx}{\sqrt{\frac{\alpha+\beta}{2} + \frac{(\alpha-\beta)x}{1+x^2}}} \tag{3.21}$$

$$z_o = \frac{Q}{\pi V(A)} \int_{s_1}^{s} \frac{\operatorname{Tan} s \exp\,(i\omega(1,s))}{\{(\frac{\alpha+\beta-2}{\alpha-\beta}) \cos s+1\}\{\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2\cos s}\}^{\frac{1}{2}}} \quad .$$

When $x_1 = \delta$ and $x \to +1$ then the equation of the surface (S) is obtained. When $x_1 = -1$ and $x \to \delta$ then the equation of the surface (S_1) is obtained. When $s_1 = 0$ and $s \to \pi/2$ the equation of the streamline (L_1) is obtained. When $s_1 = \pi/2$ and $s \to \gamma$ then the equation of the equipotential CC' is obtained. When $s_1 = \gamma$ and $s \to \pi$ the equation of the free streamline (L) is obtained.

We can use the Cauchy Riemann equations for the z plane to write:

$$\frac{\partial f}{\partial r}(1,s) = \frac{\partial f}{\partial n_e}(1,s) = \frac{\partial g}{\partial s}(1,s) = g'(1,s) \quad ,$$

where n_e is the outward unit normal vector to the region and in this case coincides with the radius.

The problem as it is now presented is such that on one part of the boundary the real part of $\omega(z)$ is given and on the other part the outward normal derivative of the real part is given as a function of the real and imaginary parts of $\omega(z)$. This problem has been studied by Poncin and Villat [1] and [2].

The method of successive approximations is used to solve the problem. The solution we seek will be represented as the limit of a sequence of functions $\omega_n(z)$ such that

$$\omega(z) = \lim_{n \to \infty} \omega_n(z) ,$$

where $\omega_n(z)$ represents a function regular in the upper unit semi-circle defined by the following boundary conditions:

$$\omega_n(\delta) = 0 , \quad -1 < \delta = \frac{\alpha-\beta-2\sqrt{(1-\beta)(\alpha-1)}}{2-\alpha-\beta} < 0 ,$$

$$f_n(x,0) = \theta(x) \quad \text{on the diameter} \quad -1 \leq x \leq +1 , \tag{3.22}$$

$$\frac{df_n}{dr}(1,s) = +\mu R_1(s) \sin f_{n-1}(1,s) e^{-3g_{n-1}(1,s)} \quad \text{on the circumference} \quad \pi \geqq s \geqq 0 .$$

Here $R_1(s)$ is defined in (3.19) and we shall assume that

$$\theta(x) = \begin{cases} 0 \quad \text{for} \quad -1 \leqq x \leqq \delta < 0 , \\ \theta_o(x)(x-\delta) \quad \text{as} \quad x \to \delta^+ \quad \text{for} \quad |\theta_o(x)| < \frac{\pi}{4} \quad \text{in some neighbourhood of} \quad x = \delta . \end{cases}$$

First Approximation

In solving this problem we choose the first approximation to be gravity free and hence $g = 0$. Therefore the function $\omega_1(z)$ must be regular in the unit semi-circle and satisfy:

$$\omega_1(\delta) = 0 ,$$

$$f_1(x,0) = \theta(x) \quad \text{on the diameter} \quad -1 \leqq x \leqq 1 ,$$

$$\frac{df_1}{dr}(1,s) = 0 \quad \text{on the circumference} \quad 0 \leqq s \leqq \pi \ .$$

The bounded solution to this problem is known and may be found for example in Poncin [2] to be

$$\omega_1(z) = \frac{i}{\pi}\int_{-1}^{+1} \theta(x) \left\{ \frac{z^2-1}{(x-z)(1-xz)} - \frac{(\delta^2-1)}{(x-\delta)(1-\delta x)} \right\} dx \ . \qquad (3.23)$$

<u>Approximation to order n</u>

We construct the new function

$$\Omega_n(z) = F_n(x,y) + iG_n(x,y) = \omega_n(z) - \omega_1(z)$$

which is regular in the unit semi-circle and satisfies the following boundary conditions.

$$\Omega_n(\delta) = 0 \qquad -1 \leq \delta \leq 0$$

$$F_n(x,0) = 0 \quad \text{on the diameter} \quad -1 \leqq x \leqq +1 \qquad (3.24)$$

$$\frac{dF_n}{dr}(1,s) = +\mu R_1(s) \sin f_{n-1}(1,s) \exp\left(-3g_{n-1}(1,s)\right)$$

$$\text{on the circumference} \quad 0 \leqq s \leqq \pi$$

where $R_1(s)$ is defined by (3.19).

The solution to this problem may be found for example in H. Villat [1] or Poncin [2] and it is

$$\Omega_n(z) = \frac{\mu}{\pi}\int_0^{\pi} \sin f_{n-1}(1,s) e^{-3g_{n-1}(1,s)} R_1(s) \operatorname{Log}\left\{\frac{e^{is}-z}{e^{-is}-z} \times \frac{e^{-is}-\delta}{e^{is}-\delta}\right\} ds \ . \quad (3.25)$$

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We will investigate the behavior of $F_n(x,y)$ and $G_n(x,y)$ on the boundary of the unit semi-circle but we note that for $z = \exp iu$ we obtain

$$\operatorname{Im}\left\{\operatorname{Log}\frac{e^{is}-z}{e^{-is}-z}\right\} = \rho(u,s) = \begin{cases} \pi+s & \text{for } u < s \\ s & \text{for } u > s \end{cases} , \tag{3.26}$$

On the boundary of the upper unit semi-circle for $0 \leq s \leq \pi$ on the circumference and $-1 \leq x \leq 1$ on the diameter the real and immaginary parts become:

$$f_1(1,s) = \frac{2\sin s}{\pi}\int_{-1}^{+1}\frac{\theta(u)}{u^2-2u\cos s+1}\,du ,$$

$$g_1(1,s) = \frac{1}{\pi}\int_{-1}^{+1}\frac{\theta(u)(1-\delta^2)}{(u-\delta)(1-u\delta)}\,du = \text{constant} ,$$

$$g_1(x,0) = \frac{1}{\pi}\int_{-1}^{+1}\theta(u)\left\{\frac{(x^2-1)}{(u-x)(1-ux)} - \frac{(\delta^2-1)}{(u-\delta)(1-u\delta)}\right\}du ,$$

$$f_1(x,0) = \theta(x) , \tag{3.27}$$

$$F_n(1,s) = \frac{\mu}{\pi}\int_0^{\pi}\sin f_{n-1}(1,u)e^{-3g_{n-1}(1,u)}R_1(u)\operatorname{Log}\left|\frac{\sin(u+s)/2}{\sin(u-s)/2}\right|du ,$$

$$G_n(1,s) = \frac{\mu}{\pi}\int_0^{\pi}\sin f_{n-1}(1,u)e^{-3g_{n-1}(1,u)}R_1(u)\{\rho(s,u)-2\operatorname{Arctan}\{\frac{\sin u}{\cos u-\delta}\}\}du ,$$

$$G_n(x,0) = \frac{\mu}{\pi}\int_0^{\pi}\sin f_{n-1}(1,u)e^{-3g_{n-1}(1,u)}2R_1(u)\operatorname{Arctan}\left(\frac{1+x}{1-x}\operatorname{Tan}\frac{u}{2}\right)du ,$$

$$F_n(x,0) = 0 .$$

CHAPTER IV

STUDY OF SOLUTIONS OF THE PROBLEM

We will study the two cases $\beta = 1$ and $\beta < 1$ with $\theta(x) = 0$ for $-1 \leq x \leq \delta$. In the latter case the uniform convergence and uniform boundedness follow quite easily since the functions are known and well behaved in a vicinity of $z = -1$. Otherwise the analysis is more complicated as we shall see. The case $\beta = 1$ introduces a singularity in the integrand which demands special attention. In most cases the lower surface (S) will become horizontal after some point a short distance upstream, so the fluid surface (L) will become horizontal also at a finite distance upstream of this point for the case $\beta = 1$. We shall assume since the fluid is inviscid that the flow is not affected by imposing a flat surface (S_1) on the upper surface of the fluid a great distance upstream. This will correspond to the case $\beta < 1$ and will eleminate the necessity of treating a singularity in the integrand.

Apart from a constant, $\omega_1(z)$ is the same for both cases so we will analyse it first.

As previously mentioned we shall follow the technique of Poncin. We shall however, postulate different conditions on θ and use an alternative argument. As we mentioned in chapter two, Poncin considers the very general case of when θ is bounded and measurable. Our technique is somewhat restrictive but realistic in the physical sense. Then too, the kernels considered here are quite different from those of Poncin and require careful investigation.

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Study of $\omega_1(z)$

Clearly it suffices to study the case $\beta = 1$. Then

$$\frac{i}{\pi} \int_{-1}^{+1} \frac{\theta(x)(z^2-1)}{(x-z)(1-xz)}\, dx \quad , \tag{4.1}$$

is analytic outside the interval $[-1,+1]$ of the real axis. We shall assume that

1) $\theta(x)$ is Hölder continuous on $[-1,+1]$ except at a finite number of points where discontinuities of the first kind occur. (θ is Hölder continuous in D' if for arbitrary x' and x'' in D' , $|\theta(x')-\theta(x'')| \leqq A|x'-x''|^{\zeta}$ for A and $\zeta < 1$ positive constants.)

2) $\theta(x) = (1+x)\theta_o(x)$ for $|\theta_o(x)| \leqq \frac{\pi}{4}$, (4.2)

3) $|\theta(x)-\theta(1)| \leqq \lambda|x-1|$ for $1-\epsilon_1 \leqq x \leqq 1$, where λ is constant and ϵ_1 is a small number.

Now the equation

$$\frac{1}{\pi} \int_{-1}^{+1} \frac{\theta(x)(u^2-1)}{(x-u)(1-ux)}\, dx \quad , \tag{4.3}$$

must be interpreted in the sense of Cauchy Principal values. Clearly, however, if $\theta(x)$ is Hölder continuous in a vicinity of the point u then the Principal Value of (4.3) exists (see [6]) and is found in terms of the integral in the ordinary sense. However if $\theta(x)$ presents any jump discontinuities then $\omega(z)$ will have logarithmic singularities at

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these points. For example if (S) presents an angle pointing into the stream or away from the stream domain then the velocity at this point becomes infinite and zero respectively.

Due to the analyticity of $\omega_1(z)$ outside of $[-1,+1]$, $f_1(1,s)$ is differentiable on $(0,\pi)$ and if $|\theta(x)| \leq 2M\pi$ then

$$|f_1(1,s)| \leq 2M\pi \quad \text{for} \quad 0 \leq s \leq \pi \quad , \tag{4.4}$$

because

$$\frac{2 \sin s}{\pi} \int_{-1}^{+1} \frac{dx}{(x - \cos s)^2 + \mathrm{Sin}^2 s} = 1 \quad .$$

Now how does $f_1(1,s)$ behave as $s \to \pi$ and $s \to 0$?

$$|f_1(1,s)| \leq \frac{2 \sin s}{\pi} \int_{-1}^{+1} \left| \frac{(x+1)\ \theta_o(x)}{(x - \cos s)^2 + \mathrm{Sin}^2 s} \right| dx$$

$$\leq 2M \left\{\frac{\mathrm{Sin}\ s}{2} \mathrm{Log}\ (x - \mathrm{Cos}\ s)^2 + (\mathrm{Sin}^2\ s)\Big|_{-1}^{+1} + \frac{\pi}{2}(1 + \mathrm{Cos}\ s)\right\}$$

$$= M\ \mathrm{Sin}\ s\ \mathrm{Log}\ \left\{\frac{1 - \mathrm{Cos}\ s}{1 + \mathrm{Cos}\ s}\right\} + M\pi\ (1 + \mathrm{Cos}\ s)$$

$$\leq k_o(\pi - s)^{1-\epsilon_2} \quad ,$$

for constants k_o and $\epsilon_2 > 0$ but small. Hence

$$f_1(1,s) \to 0 \quad \text{as} \quad s \to \pi, \tag{4.5}$$

Using (4.4) we can write

$$|f_1(1,s) - \theta(1)| \leqq \frac{2 \sin s}{\pi} \int_{-1}^{+1} \frac{|\theta(x) - \theta(1)|}{(x - \operatorname{Cos} s)^2 + (\operatorname{Sin} s)^2} \, dx$$

$$= \frac{2 \operatorname{Sin} s}{\pi} \left\{ \int_{-1}^{1-\epsilon_1} \frac{|\theta(x)-\theta(1)|}{(x - \operatorname{Cos} s)^2 + (\operatorname{Sin} s)^2} dx + \int_{1-\epsilon_1}^{+1} \frac{|\theta(x)-\theta(1)|}{(x - \operatorname{Cos} s)^2 + (\operatorname{Sin} s)^2} dx \right. ,$$

where $\epsilon_1 > 0$ and (4.2) part 4 is satisfied. Then

$$\frac{2 \operatorname{Sin} s}{\pi} \int_{-1}^{1-\epsilon_1} \frac{|\theta(x)-\theta(1)|}{(x - \operatorname{Cos} s)^2 + \operatorname{Sin}^2 s} \, dx \leq h|s| \quad ,$$

for some $h > 0$.

Also

$$\frac{2 \operatorname{Sin} s}{\pi} \int_{1-\epsilon_1}^{1} \frac{|\theta(x)-\theta(1)|}{(x - \operatorname{Cos} s)^2 + \operatorname{Sin}^2 s} \, dx \leqq \frac{2\lambda \sin s}{\pi} \int_{1-\epsilon_1}^{1} \frac{(1-x)dx}{(x - \operatorname{Cos} s)^2 + \operatorname{Sin}^2 s} \, dx$$

$$= \frac{2\lambda \operatorname{Sin} s}{\pi} \left\{ -\frac{1}{2} \operatorname{Log} \left| \frac{(1 - \operatorname{Cos} s)^2 + \operatorname{Sin}^2 s}{(1-\epsilon_1 - \operatorname{Cos} s)^2 + \operatorname{Sin}^2 s} \right| \right\} +$$

$$+ (1 - \operatorname{Cos} s) \frac{2\lambda}{\pi} \operatorname{Arctan} \frac{\epsilon_1 \operatorname{Sin} s}{(\epsilon_1 - 2)(\operatorname{Cos} s - 1)} \quad .$$

Hence

$$|f_1(1,s)-\theta(1)| \to 0 \quad \text{as} \quad s \to 0 \quad , \qquad \text{and}$$

$$f_1(1,s) \to \theta(1) \quad \text{as} \quad s \to 0 \quad . \qquad (4.6)$$

Near $s = \pi$ we have the expression

$$|f_1(1,s)| \leq k_0(\pi-s),^{1-\epsilon_2}$$

while on $(0,\pi)$, $f_1(1,s)$ is differentiable and continuous and

$$|f_1(1,s)| \leq 2M\pi \quad \text{for} \quad 0 \leq s \leq \pi \quad .$$

Then there exists a constant k_1 such that

$$|f_1(1,s)| \leq k_1 |\pi-s|^{1-\epsilon_2} \quad \text{for} \quad 0 \leq s \leq \pi \,. \tag{4.7}$$

Study of $\Omega_n(z)$

Now the real part of the function Ω_n is zero on the diameter of the unit semi-circle of the z plane and Ω_n is analytic in its interior. The function $W_n = -i\Omega_n$ whose imaginary part is zero on the real axis and analytic inside the upper unit semi-circle can be continued analytically into the lower unit semi-circle by defining it there to be $\bar{W}(\bar{z})$. In this way Ω_n is also extended to the whole circle $|z| < 1$ and is analytic at all points of this domain. Hence we will refer to function $\Omega_n(z)$ either to the unit circle $|z| = 1$ or just the upper unit semi-circle whenever it is convenient to do so. Let $D=\{z \,|\, |z| \leqq 1\}$.

Case 1. $\beta < 1$

The expression $R_1(s)$ occurs in the function $\Omega_n(z)$. It is defined by

$$R_1(s) = \begin{cases} 0 & \text{for } \gamma > s > \frac{\pi}{2}, \\ \dfrac{-\text{Tan } s}{\left(\frac{\alpha+\beta-2}{\alpha-\beta}\text{ Cos } s+1\right)\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2\text{ Cos } s}\right)^{\frac{1}{2}}} & \text{for } \pi > s > \gamma, \\ & \frac{\pi}{2} > s > 0. \end{cases}$$

The singularities at $s = \gamma$ and $s = \frac{\pi}{2}$ are square root singularities and hence are integrable. If $\beta < 1$ and $\alpha > 1$ as is the case here no other singularities occur in $R_1(s)$. In the integrand of $\Omega_n(z)$ a singularity occurs at $s = u$ but this too is integrable since it is a logarithmic singularity. It can be shown that under the conditions of (4.2) on $\theta(x)$ and the boundedness of $\Omega_{n-1}(z)$ on the circumference of the circle $|z| = 1$ that $\Omega_n(z)$ is continuous there too. (See the appendix.)

Using the above facts $f_n(1,s)$ and $g_n(1,s)$ will be shown to be uniformly bounded on the circumference of the circle $|z| = 1$. Suppose that $\omega_1(z)$ is defined in the lower unit semi-circle by $\bar{\omega}_1(\bar{z})$ where z is in this domain. Since the singularities in ω_1 occur on the diameter of the upper unit semi-circle and not on the circumference $f_1(1,s)$ and $g_1(1,s)$ are bounded on the circumference of the circle $|z| = 1$. Let the least upper bound be denoted by F_1 and G_1 respectively for F_1 and G_1 positive numbers. Let F_n and G_n be upper bounds for the functions $f_n(1,s)$ and $g_n(1,s)$ on the circumference of the circle $|z| = 1$. The integrals in (3.27) all exist since the singularities are of order less than one of the integrand. Then

$$|t_n(1,s)| \leqq F_1 + \frac{\mu}{\pi} e^{3G_{n-1}} \int_0^{\pi} |R_1(u) \text{ Log } | \frac{\text{Sin } (u+s)/2}{\text{Sin } (u-s)/2} || \, du \quad ,$$

$$|g_1(1,s)| \leqq G_1 + \frac{\mu}{\pi} e^{3G_{n-1}} \int_0^{\pi} |R_1(u) \, \{\rho(u,s) - 2 \text{ Arctan } \{\frac{\text{Sin } u}{\text{Cos } u-\delta}\}\}| \, du \quad .$$

Using the same argument as used in proving $\omega_n(z)$ continuous (see appendix) we can prove that the integrals above are continuous on the circumference of the unit semi-circle. Here though $f_{n-1}(1,s) = \frac{\pi}{2}$ and $g_{n-1}(1,s) = 0$. Hence we can take the maximum value of these two integrals as s traverses the circumference of the semi-circle. Call these maxima FF and GG respectively and define F_n and G_n such that

$$F_n = F_1 + \frac{\mu}{\pi} e^{3G_{n-1}} FF \tag{4.8a}$$

$$G_n = G_1 + \frac{\mu}{\pi} e^{3G_{n-1}} GG \quad . \tag{4.8b}$$

Clearly $G_2 \geqq G_1$ and by the mathematical induction it follows that $G_{n+1} \geqq G_n$ so the $\{G_n\}$ are monotonically increasing sequences of positive numbers. If these sequences are bounded above the least upper bounds are respectively denoted by F and G where F and G are solutions to the equations

$$F = F_1 + \frac{\mu}{\pi} e^{3G} FF \tag{4.8c}$$

$$G = G_1 + \frac{\mu}{\pi} e^{3G} GG \quad . \tag{4.8d}$$

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Suppose that we can determine a value of the parameter μ such that equation (4.8d) is satisfied by two values of G which we denote by G_L and G_U where $G_L < G_U$. Since for $G > 0$

$$G_1 < G_1 + GG \frac{\mu}{\pi} \exp 3G ,$$

then

$$G_1 < G_L ,$$

which is necessary since the sequence $\{G_n\}$ is monotonically increasing. Suppose for some k, G_k is such that $G_U > G_k > G_L$. Then it is true that

$$G_k > G_1 + \frac{\mu}{\pi} GG\, e^{3G_k} = G_{k+1} ,$$

which contradicts the fact that the sequence is increasing. Hence G_L is an upper bound for the sequence $\{G_n\}$ and the sequences $\{f_n(1,s)\}$ and $\{g_n(1,s)\}$ are uniformly bounded on the boundary of the circle $|z| = 1$ providing a solution to (4.8d) exists, i.e.,

$$\mu < \frac{\pi}{3(GG) \exp (3G+1)} . \tag{4.9a}$$

The value of μ was obtained as follows. Let

$$y_1 = G,$$

$$y_2 = G_1 + \frac{\mu}{\pi} GG \exp 3G,$$

and let us determine μ such that these two equations are satisfied by only one value. This makes μ as large as possible. Then y_1 must

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be tangent to the curve y_2 at some point. Here the slope is unity so

$$\frac{dy_2}{dG} = 1 = \frac{3\mu}{\pi} (GG) \exp 3G$$

and hence

$$G = \frac{1}{3} \text{Log} \frac{\pi}{3\mu(GG)} \quad .$$

When this value of G is substituted in (4.8d) an expression similar to (4.9a) is obtained but with equality.

The sequences $\{F_n(1,s)\}$, $\{G_n(1,s)\}$ and hence $\{\Omega_n(z)\}$ are uniformly bounded on the circumference of the circle $|z| = 1$. Clearly the functions $\Omega_n(z)$ are analytic in the interior of this circle and by a theorem due to Montel [4]

Theorem - "Toute suite infinie de fonctions holomorphes bornées dans un domaine (D) est génératrice d'une suite partielle convergeant uniformément vers une limite."

we know that the sequence admits at least one limit function. It remains to find the limit function and show that it is unique and that it satisfies the boundary conditions to the problem.

Now we shall investigate convergence of the sequence above. We note that since $|\Omega_n - \Omega_{n-1}| = |\omega_n - \omega_{n-1}|$, it suffices to study the former. Note too that

$$|\sin f_n(1,u)e^{-3g_n(1,u)} - \sin f_{n-1}(1,u)e^{-3g_{n-1}(1,u)}|$$

$$\leqq |\sin f_n(e^{-3g_n} - e^{-3g_{n-1}}) + e^{-3g_{n-1}}(\sin f_n - \sin f_{n-1})|$$

$$\leqq e^{3G}(3|g_n - g_{n-1}| + |f_n - f_{n-1}|) \ , \qquad (4.9b)$$

where G is defined above. Hence

$$|(F_3-F_2)(1,s)| \leqq \frac{\mu}{\pi} e^{3G} \int_0^{\pi} |R_1(u) \operatorname{Log} \left|\frac{\operatorname{Sin}(u+s)/2}{\operatorname{Sin}(u-s)/2}\right| \cdot (3|G_2-G_1| + |F_2-F_1|)|du$$

$$\leqq \frac{\mu}{\pi} e^{3G} \cdot 4 \cdot L \cdot FF \ ,$$

where $L = \text{Maximum } \{ |G_2-G_1|, |F_2-F_1| \}$,

Similarly

$$|(G_3-G_2)(1,s)| \leq \frac{\mu}{\pi} e^{3G} \cdot 4 \cdot L \cdot GG$$

and thus assume that

$$|F_n - F_{n-1}| \leq (4 \frac{\mu}{\pi} e^{3G} M)^{n-2} = e_*^{n-2} \ ,$$

$$|G_n - G_{n-1}| \leq (4 \frac{\mu}{\pi} e^{3G} M)^{n-2} = e_*^{n-2} \ ,$$

for all $n < N$ for N given and M/L is the Maximum of FF and GG . It follows easily that

$$|F_N - F_{N-1}| \leq e_*^{N-2}$$

and

$$|G_N - G_{N-1}| \leq e_*^{N-2} \ .$$

Uniform convergence of these sequences will follow providing $e_* < 1$ and hence

$$\mu < \frac{\pi}{4M} e^{-3G} \quad . \tag{4.10}$$

So providing (4.10) is satisfied by μ the sequences $\{G_n(1,s)\}$ and $\{F_n(1,s)\}$ are uniformly convergent to functions $G(1,s)$ and $F(1,s)$ which are continuous on the circumference of the circle $|z| = 1$ since $F_n(1,s)$ and $G_n(1,s)$ are continuous there. Now let $\Omega^1(z)$ and $\Omega^2(z)$ be two limit functions of the sequence $\{\Omega_n(z)\}$. Then these two functions are analytic in the domain D because each $\Omega_n(z)$ is analytic there and the sequence is uniformly convergent. Then for $k = 1,2$ we can write

$$\Omega^k(z) = \lim_{n\to\infty} \sum_{i=2}^{n} (\Omega_i - \Omega_{i-1}) + \Omega_1 \quad ,$$

so for $n > m$ in D , we have (n corresponds to k=1 and m to k=2)

$$|\Omega^1(z) - \Omega^2(z)| \leq \lim_{n,m\to\infty} \sum_{m+1}^{n} |\Omega_i - \Omega_{i-1}| \quad .$$

On the boundary of D we can write

$$|\Omega^1 - \Omega^2| \leq \lim_{n,m\to\infty} 2 \sum_{m+1}^{n} e_*^{i-2} = 0 \quad .$$

Since $\Omega^1(z) - \Omega^2(z)$ is analytic throughout D and is zero on the boundary of D it must be zero for all z in the domain D by the

Maximum Principle. Hence $\Omega^1(z)$ and $\Omega^2(z)$ are identical and the limit function is unique.

Are the boundary conditions of the problem satisfied by this limit function. Clearly the conditions one and two of (3.24) are satisfied. From (3.27) and (3.19) we can write

$$G_n(1,s) = +\frac{\mu}{\pi}\int_{\pi}^{s} \sin f_{n-1}(1,u)e^{-3g_{n-1}(1,u)} R_1(u)du + c \quad .$$

Due to the uniform convergence of the sequence $\{f_n(1,s)\}$ and $\{g_n(1,s)\}$ on the circumference of the circle $|z| = 1$ we can write

$$\underset{n\to\infty}{\text{Limit}}\; G_n(1,s) = +\frac{\mu}{\pi}\int_{\pi}^{s} \underset{n\to\infty}{\text{limit}}\,(\sin f_{n-1}(1,s)e^{-3g_{n-1}(1,s)})R_1(s)ds + c$$

$$= G(1,s) = +\frac{\mu}{\pi}\int_{\pi}^{s} \sin f(1,u)e^{-3g(1,u)}R_1(u)du + c \;.$$

Therefore on differentiating this expression we obtain

$$G'(1,s) = +\frac{\mu}{\pi}\sin f(1,s)e^{-3g(1,s)}R_1(s) \;,$$

which is the third boundary condition of (3.14) or equivalently (3.25).

Hence $\omega(z)$ is the unique limit to the sequence of successive approximations and satisfies the boundary conditions.

<u>Case 2. $\beta = 1$</u>.

With the following notation

$$R_1(u) = \begin{cases} 0 & \text{for } \frac{\pi}{2} < u < \gamma \quad , \\ \dfrac{-\operatorname{Tan} u}{1+\cos u}\left(\dfrac{\alpha+1}{2} + \dfrac{\alpha-1}{2\cos u}\right)^{\frac{1}{2}} & \text{for } \begin{cases} 0 < u < \frac{\pi}{2} \\ \gamma < u < \pi \end{cases} , \end{cases} \tag{4.11}$$

$$\Phi_n(u) = \sin f_{n-1}(1,u)\, e^{-3g_{n-1}(1,u)}\, R_1(u) \quad ,$$

the real and imaginary parts of the sequence $\{\Omega_n(z)\}$ can be written in the following manner on the boundary of the upper unit semi-circle.

$$F_n(1,s) = \frac{\mu}{\pi}\int_0^{\pi} \Phi_n(u)\, \operatorname{Log}\left|\frac{\sin (u+s)/2}{\sin (u-s)/2}\right| du \quad , \tag{4.12}$$

$$G_n(1,s) = \mu\int_{\pi}^{s} \Phi_n(u)\, du$$

One may proceed as indicated in the case $\beta < 1$ using an adaption of a technique developed by Poncin [2]. The difficulty here is the singularity of $R_1(u)$ at $u = +\pi$. We shall attempt an alternative proof below based on a theorem by Golusin [7].

Theorem: Die Funktion $f(z) = u(r,\theta) + iv(r,\theta)$ sei in $|z| < 1$ regulär; ferner sei $u(r,\theta)$ in $|z| \leqq 1$ stetig und genuge auf $|z| = +1$ der Lipschitz-Bedingung

$$|u(\theta)-u(\theta')| \leqq k|\theta-\theta'|^{\alpha} , \quad 0 < \alpha < 1 \quad .$$

Dann genügt $f(z)$ in $|z| \leqq 1$ einer komplexen Lipschitz-Bedingung

$$|f(z)-f(z')| < K\,|z-z'|^{\alpha} \quad .$$

Here the constant K is dependent on the constants k and α .

The problem as we have set it up is to determine a sequence of functions $\{\Omega_n(z)\}$ which is regular in the upper unit semi-circle and satisfies the following boundary conditions.

$$\Omega_n(-1) = 0 ,$$

$$\frac{\partial G_n(1,s)}{\partial s} = +\mu\Phi_n(s) \text{ on the circumference of the semi-circle,}$$

$$F_n(x,0) = 0 \text{ on the diameter of the semi-circle.}$$

We shall deal with a different formulation of the problem and pose the question of determining a sequence of functions $W_n(z)$ regular in the interior of the circle $|z| = 1$ such that $W_n(-1) = 0$ and is specified by the boundary conditions

$$G_n(1,s) = \mu \int_{\pi}^{s} \Phi_n(u)du \quad \text{for} \quad 0 \leqq s \leqq \pi ,$$

$$G_n(1,s) = G_n(1, 2\pi - s) \quad \text{for} \quad \pi \leqq s \leqq 2\pi \quad . \tag{4.12}$$

Clearly $W_n(z)$ coincides with $-i\Omega_n(z)$ in the upper semi-circle. We must prove that G_n , the real part of W_n , is continuous in $|z| \leqq 1$ and satisfies a Lipschitz condition on $|z| = 1$ in order to use the theorem of Golusin stated above. It is not difficult to show that G satisfies these conditions but we shall illustrate the procedure by proving that G_n satisfies a Lipschitz condition on the circumference $|z| = 1$. Once this is done it follows in particular that

$$|f(e^{i\theta_1})-f(e^{i\theta_2})| \leqq K|\theta_1-\theta_2|^{\alpha} \quad \text{for} \quad 0 < \alpha < 1 \ ,$$

where $0 \leq \theta_1,\theta_2 \leq 2\pi$ and $K = k\sigma(\alpha)$. (See Golusin).

We shall first show that for $0 \leqq s_1,s_2 \leq \pi$

$$|G_2(1,s_1)-G_2(1,s_2)| \leqq k|s_1-s_2|^{\frac{1}{2}} \quad . \tag{4.14}$$

The extension to the interval $[0,2\pi]$ then follows by symmetry. By setting $s_2 = \pi$ and noting that $G_2(1,s) = 0$ at $s = \pi$ it follows that $G_2(1,s)$ is bounded and continuous for $0 \leq s \leq 2\pi$. From previous considerations such as expression (4.7) with constants k_1 and $\epsilon_2 = 1/2$ we have

$$g_1(1,s) = 0 \quad ,$$

$$|f_1(1,s)| \leqq k_1|\pi-s|^{\frac{1}{2}}$$

for $0 \leq s \leq \pi$ and hence $f_1(1,s)$ is a bounded continuous function on the interval $[0,\pi]$.

To prove (4.14) we shall break the discussion into two parts:

$$I_1 \qquad \gamma < \delta_1 \leqq s \leqq \pi \quad ,$$

$$I_2 \qquad 0 \leqq s \leq \delta_1^* \ \text{where} \ \pi > \delta_1^* > \delta_1 \quad .$$

With the form of $g_1(1,s)$ above it follows that $G_2(1,\pi) = 0$. In the region I , it follows that

$$
\begin{aligned}
|G_2(1,s_1)-G_2(1,s_2)| &= \mu \left| \int_{s_1}^{s_2} \sin f_{n-1}(1,u) R_1(u)\,du \right| \\
&\leqq \mu k_1 \int_{s_1}^{s_2} \left| \sqrt{(\pi-s)}\, R_1(s) \right| ds \\
&\leqq \mu k_1 h_1 \int_{s_1}^{s_2} \frac{ds}{\sqrt{\pi-s}} \\
&\leqq \mu\, k_1 h_1 \int_{0}^{s_2-s_1} \frac{dv}{\sqrt{v}} \quad , \\
&= 2\mu k_1 h_1 \, |s_2-s_1|^{\frac{1}{2}} \quad ,
\end{aligned}
$$

where $s_2 > s_1$ and

$$
h_1 = \underset{I_1}{\text{Maximum}} \left| \left\{ \frac{\sin s}{1+\cos s} \times \frac{\pi-s}{\cos s} \times \left(\frac{\alpha+1}{2} + \frac{\alpha-1}{2\cos s}\right)^{-\frac{1}{2}} \right\} \right| .
$$

Now consider the interval I_2 and denote the cosed intervals bounded by the pairs $[0, \pi/2]$, $[\pi/2, \gamma]$, and $[\gamma, \pi]$ by L_1, L_2 and L_3 respectively.

If s_1 and s_2 are in L_2 the result is trivial since the integrand is null.

$$
|G_2(1,s_1)-G_2(1,s_2)| = 0 \leqq |s_1-s_2|^{\frac{1}{2}} \quad .
$$

If s_1 and s_2 are in L_1 then

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$$|G_2(1,s_1)-G_2(1,s_2)| \le \mu h_2 \left| \int_{s_1}^{s_2} \frac{du}{\sqrt{\frac{\pi}{2} - u}} \right| k_1$$

$$\leqq 2\mu h_2 \, |s_2-s_1|^{\frac{1}{2}} k_1 \ ,$$

where

$$h_2 = \underset{L_1}{\text{Maximum}} \left\{ \frac{\sin s \ \sqrt{\pi}}{1 + \cos s} \sqrt{\left| \frac{(\frac{\pi}{2} - s)}{\cos s} \left(\frac{\alpha+1}{2} \cos s + \frac{\alpha-1}{2}\right)^{-1} \right|} \right\} \ .$$

If s_1 and s_2 are in L_3 the procedure is the same and

$$|G_2(1,s_1)-G_2(1,s_2)| \le 2\mu h_3 |s_2-s_1|^{\frac{1}{2}} k_1 \ .$$

If s_1 is in L_1 and s_2 is in L_2 then

$$|G_2(1,s_1)-G_2(1,s_2)| = \mu \left| \int_{s_1}^{\pi/2} \Phi_2(u)du \right|$$

$$\leq 2\mu h_2 \left|\frac{\pi}{2} - s_1\right|^{\frac{1}{2}} k_1$$

$$\leq 2\mu h_2 \, |s_2-s_1|^{\frac{1}{2}} k_1 \ .$$

If s_1 and s_2 are in L_2 and L_3 respectively, then as above there exists a constant h_3 such that

$$|G_2(1,s_1)-G_2(1,s_2)| \leqq 2\mu h_3 \, |s_2-s_1|^{\frac{1}{2}} k_1 \ .$$

If s_1 and s_2 are in L_1 and L_3 respectively, then by

a combination of the arguments above there exists a constant h_4 such that

$$|G_2(1,s_1)-G_2(1,s_2)| \leq 2\mu h_4 k_1 \; |s_1-s_2|^{\frac{1}{2}} \quad .$$

Clearly then, since the regions I_1 and I_2 overlap we can find a number T which depends only in the integral above and is independent of the subscript n such that for $0 \leq s_1 \leq s_2 \leq \pi$

$$|G_2(1,s_1)-G_2(1,s_2)| \leq \mu k_1 T \; |s_1-s_2|^{\frac{1}{2}} \quad . \tag{4.16}$$

It is always possible to determine T and k_1 although it may sometimes be difficult. We may now apply the theorem of Golusin and obtain the result:

$$|\Omega_2(z_1)-\Omega_2(z_2)| \leq \mu k_1 T\sigma(\tfrac{1}{2}) \; |z_1-z_2|^{\frac{1}{2}} \quad ,$$

for $|z_1| \leq 1$ and $|z_2| \leq 1$. Also $\sigma = \sigma(\frac{1}{2})$ is independent of the subscript n which in this case is 2 . Since $G_2(1,\pi) = 0$ and F_2 is zero on the diameter it follows that $\Omega_2(-1) = 0$. In particular

$$|\Omega_2(e^{is_1}) - \Omega_2(e^{is_2})| \leqq \mu k_1 T\sigma \; |s_1-s_2|^{\frac{1}{2}} \quad ,$$

for $0 \leqq s_1,s_2 \leq \pi$. Now choose μ such that for some $D > 0$

$$\mu(1+\sigma D)T \exp 3k_1 \; D\sqrt{\pi} < D \quad .$$

For such a μ and D it follows that

$$\mu T < D \quad .$$

In order to keep μ as large as possible one would then maximize the quotient

$$\frac{D}{(1+\sigma D)T\exp(3k_1 D\sqrt{\pi})} \tag{4.17}$$

to determine D. We deduce that

$$|F_2(1,s_1)-F_2(1,s_2)| \leq k_1\sigma D\ |s_1-s_2|^{\frac{1}{2}}$$

$$|G_2(1,s_1)-G_2(1,s_2)| \leq k_1\sigma D\ |s_1-s_2|^{\frac{1}{2}} \quad .$$

Now set $s_2 = \pi$ and $s_1 = s$ so that with our previous discussion on $f_1(1,s)$ and $g_1(1,s)$ we can write for $0 \leq s \leq \pi$

$$\begin{aligned} |f_2(1,s)| &\leq k_1(1+\sigma D)(\pi-s)^{\frac{1}{2}} \leq k_1(1+\sigma D)\sqrt{\pi} \quad , \\ |g_2(1,s)| &\leq k_1 D\ (\pi-s)^{\frac{1}{2}} \leq k_1\ D\sqrt{\pi} \quad , \end{aligned} \tag{4.18}$$

so $f_2(1,s)$ and $g_2(1,s)$ are bounded and continuous on $0 \leq s \leq \pi$. We now repeat the argument above for $G_3(1,s)$.

In the same way as above we show that

$$\begin{aligned} |G_3(1,s_1)-G_3(1,s_2)| &\leq \mu k_1(1+\mu T\sigma)\ e^{3\mu k_1 T\sqrt{\pi}}\ T|s_2-s_1|^{\frac{1}{2}} \\ &\leq k_1 D\ |s_1-s_2|^{\frac{1}{2}} \quad . \end{aligned}$$

Again from the properties of f_2 and g_2 we deduce that $G_3(1,\pi) = 0$

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and F_3 is zero in the diameter. In the same way as above $\Omega_3(-1) = 0$.
As above from Golusin

$$|\Omega_3(e^{is_1}) - \Omega_3(e^{is_2})| \leqq k_1 D\sigma \; |s_1 - s_2|^{\frac{1}{2}}$$

and

$$|G_3(1,s_1) - G_3(1,s_2)| \leqq k_1 D \; |s_1 - s_2|^{\frac{1}{2}}$$

$$|F_3(1,s_1) - F_3(1,s_2)| \leqq k_1 D \; |s_1 - s_2|^{\frac{1}{2}} \sigma \quad .$$

In turn we obtain

$$|f_3(1,s)| \leqq k_1(1+\sigma D) \; |\pi - s|^{\frac{1}{2}} \leqq k_1(1+\sigma D) \sqrt{\pi}$$

$$|g_3(1,s)| \leqq k_1 D \; |\pi - s|^{\frac{1}{2}} \leqq k_1 D \sqrt{\pi} \quad .$$

This procedure may be repeated and hence for every n and for $0 \leqq s_1, s_2 \leqq \pi$
we have

$$|F_n(1,s_1) - F_n(1,s_2)| \leqq k_1 \sigma D \; |s_1 - s_2|^{\frac{1}{2}}$$

$$|G_n(1,s_1) - G_n(1,s_2)| \leqq k_1 D \; |s_1 - s_2|^{\frac{1}{2}}$$

$$|f_n(1,s)| \leqq k_1(1+\sigma D) \sqrt{\pi}$$

$$|g_n(1,s)| \leqq k_1 D \sqrt{\pi} \quad .$$

It follows that the sequences $\{f_n\}$ and $\{g_n\}$ as they are defined for the circle $|z| \leqq 1$ are uniformly bounded on the circumference of this circle. Hence $\{W_n(z)\}$ is uniformly bounded on the circumference of this circle but $W_n(z)$ is analytic at all points in this circle so the sequence $\{W_n(z)\}$ is uniformly bounded in the interior of the circle $|z| = 1$. By a theorem due to Montel [4] which has already been stated for the case $\beta < 1$ there exists a sub-sequence $\{W_{n_k}\}$ which is uniformly convergent to an analytic function $W(z)$. Since $W_n(z)$ coincides with $-i\Omega_n(z)$ in the upper unit semi-circle it follows that the sequence $\{\Omega_n(z)\}$ also admits a sub-sequence which converges uniformly to an analytic function. It remains to determine this limit function.

Note that the boundary values $F_n(1,s)$ and $G_n(1,s)$ are continuous functions on $[0,2\pi]$.

<u>Convergence of the Sequence $\{\Omega_n(z)\}$</u>

The proof of the convergence of the sequences $\{F_n(1,s)\}$ and $\{G_n(1,s)\}$ and hence $\{\Omega_n(z)\}$ on the boundary of the domain D is more complex for the case $\beta = 1$ than for the case $\beta < 1$ due to the addition of a singularity at $u = \pi$ in the function $R_1(u)$ which is defined in (4.11) and occurs in the integrand of the expression for F_n and G_n. This integrand is also more complex than those derived by Poncin in [2] which we discussed earlier and there is of course correspondingly more labor involved in proving uniform convergence.

In order to prove uniform convergence of the sequences involved

in this analysis we shall resort once more to the theorem of Golusin. This will enable us to attack the singularity at $u = \pi$ directly.

We shall first define a number of functions.

$$D_n(s) = G_n(1,s) - G_{n-1}(1,s) \quad \text{for } n > 2 ,$$

$$D_2(s) = G_2(1,s) , \quad v_2(z) = -i\Omega_2(z) , \tag{4.20}$$

$$v_n(z) = -i\{\Omega_n(z) - \Omega_{n-1}(z)\} \quad \text{for } n > 2 .$$

From the expression (4.16) we can write

$$|D_2(s_1) - D_2(s_2)| \leqq \mu k_1 T \, |s_2 - s_1|^{\frac{1}{2}} . \tag{4.21}$$

Since $D_n(s)$ is the real part of the analytic function $v_n(z)$ the hypothesis of the theorem of Golusin is satisfied where the functions are extended to the whole unit semi-circle as in (4.13). This yields

$$|v_2(z_1) - v_2(z_2)| \leqq \mu k_1 T\sigma \, |z_1 - z_2|^{\frac{1}{2}} .$$

With the definitions $F_1(1,s) = G_1(1,s) = 0$ and since $F_n(1,s) = G_n(1,s) = 0$ for $s = \pi$ this equation yields

$$\begin{aligned} |F_2(1,s) - F_1(1,s)| &\leqq \mu k_1 T\sigma \, |s-\pi|^{\frac{1}{2}} \\ |G_2(1,s) - G_1(1s)| &\leqq \mu k_1 \, |s-\pi|^{\frac{1}{2}} \, T . \end{aligned} \tag{4.22}$$

Let G denote the upper bound for the sequence $\{|g_n(1,s)|\}$. From

(4.9b) and the expressions (4.20) we have for $s_2 > s_1$

$$|D_3(s_1)-D_3(s_2)| \leqq \mu\, e^{3G} \int_{s_1}^{s_2} \{3|g_2-g_1| + |f_2-f_1|\}\, |R_1(u)|du \;,$$

$$\leqq \mu\, e^{3G} \int_{s_1}^{s_2} \{3|G_2| + F_2|\}\, |R_1(u)|du \;.$$

With (4.22) and using the value of T as defined in (4.16) we obtain

$$|D_3(s_1)-D_3(s_2)| \leqq \mu\, e^{3G}\, (\sigma+3)\, \mu T k_1 \int_{s_1}^{s_2} |R_1(u)(\pi-s)^{\frac{1}{2}}|du,$$

$$\leqq \mu^2 T\, e^{3G}\, (\sigma+3)\, k_1 T\, |s_1-s_2|^{\frac{1}{2}} \;, \qquad (4.23)$$

$$= (\mu T)^2\, k_1\, e^{3G}\, (\sigma+3)\, |s_1-s_2|^{\frac{1}{2}} \;.$$

The hypothesis of the theorem of Golusin is satisfied and hence

$$|v_3(z_1)-v_3(z_2)| \leqq \mu^2\, e^{3G}\, T^2 \sigma(\sigma+3)\, |z_1-z_2|^{\frac{1}{2}} \;.$$

Now set $z_2 = -1$ or $s = \pi$ and utilizing the fact $G_n(1,\pi) = F_n(1,\pi) = 0$ we write

$$|G_3(1,s)-G_2(1,s)| \leqq \mu^2\, e^{3G}\, k_1 T^2\, (\sigma+3)(\pi-s)^{\frac{1}{2}},$$

$$|F_3(1,s)-F_2(1,s)| \leqq \mu^2\, e^{3G}\, k_1 T^2\, \sigma(\sigma+3)(\pi-s)^{\frac{1}{2}} \;.$$

Proceeding as above, again we have for $s_2 > s_1$

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$$|D_4(s_1)-D_4(s_2)| \leqq \mu\, e^{3G} \int_{s_1}^{s_2} \{3|g_3-g_2| + |f_3-f_2|\}\ |R_1(u)|du$$

$$= \mu\, e^{3G} \int_{s_1}^{s_2} \{3|G_3-G_2| + |F_3-F_2|\}\ |R_1(u)|du$$

$$\leqq \mu^3\, e^{3G}\, (\sigma+3)^2\, e^{3G}\, k_1 T^2 \int_{s_1}^{s_2} |(\pi-s)^{\frac{1}{2}}\, R_1(u)|du$$

$$\leq (\mu(\sigma+3)T\, e^{3G})^2\, k_1\mu\, |s_1-s_2|^{\frac{1}{2}}\, T\ ,$$

and hence from Golusin

$$|v_4(z_1)-v_4(z_2)| \leqq \mu^3(\sigma+3)^2\, T^3\, (e^{3G})^2\, k_1\sigma\, |z_1-z_2|^{\frac{1}{2}}$$

so

$$|G_4(1,s)-G_3(1,s)| \leqq (\pi-s)^{\frac{1}{2}}\, \mu k_1\, P^2\, T,$$

$$|F_4(1,s)-F_3(1,s)| \leqq (\pi-s)^{\frac{1}{2}}\, \mu k_1\, \sigma\, P^2\, T$$

where $P = \mu(\sigma+3)T\, e^{3G}$.

Continuing in this manner we obtain for all values of the subscript $n > 2$,

$$|G_n(1,s)-G_{n-1}(1,s)| \leq P^{n-2}\, |\pi-s|^{\frac{1}{2}}\, \mu k_1 T,$$

$$|F_n(1,s)-F_{n-1}(1,s)| \leq P^{n-2}\, |\pi-s|^{\frac{1}{2}}\, \mu k_1 T\sigma \quad . \tag{4.24}$$

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Hence providing

$$\mu < \frac{1}{T(\sigma+3)} \quad e^{-3G}$$

the sequence $\{W_n(z)\}$ and hence the sequence $\{\Omega_n(z)\}$ is uniformly convergent to an analytic function.

Following the same procedure as before we know that the sequence $\{\Omega_n(z)\}$ converges to a unique limit function which is analytic in the upper unit semi-circle and satisfies the boundary conditions of the problem.

CHAPTER V

EXAMPLES

Example 1

In the paper we have referred to above [8], the theoretical results are applied to the case of a perpendicular weir with a sharp corner. Refer to the figure below.

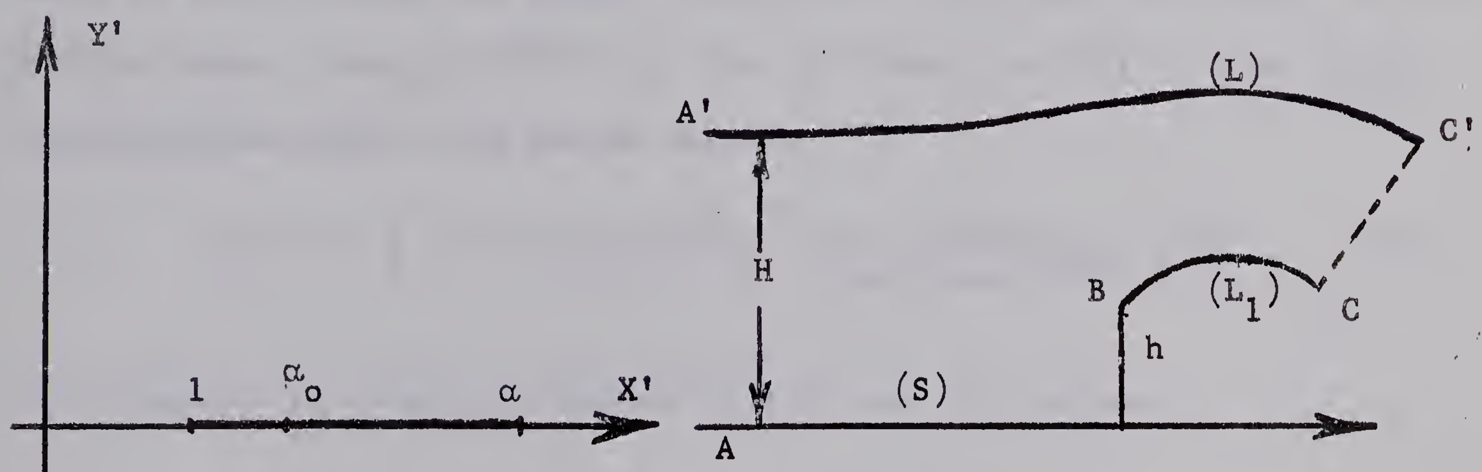

Figure 6

We shall refer to the z' plane instead of the z plane in order to compare our results with those in reference [8]. The surface (S) is represented in the z' plane by the segment $[1,\alpha]$. α and α_o in this figure refer respectively to the upper and lower edges of the perpendicular in the z_o plane. The function $\theta(x)$ may thus be written,

$$\theta(x) = \begin{cases} 0 & \text{for} \quad 1 \leqq x < \alpha_o \\ +\pi/2 & \text{for} \quad \alpha_o < x \leqq \alpha \end{cases} \tag{5.1}$$

α may be determined from expresion 3.3 and α_o is found to depend upon the height of the weir. The first appoximation of the flow is given by equation (3.23) for the upper unit semi-circle. It must be transformed under expression (3.4) to the z' plane. The mapping and its inverse are,

$$z' = \frac{\alpha+1}{2} + \frac{(\alpha-1)z}{1+z^2} ,$$

$$z = \left\{ \frac{\alpha-1}{2} + i\{(z'-\alpha)(z'-1)\}^{\frac{1}{2}} \right\} \left\{ z' - \frac{\alpha+1}{2} \right\}^{-1} , \qquad (5.2)$$

where we shall choose the branch $\operatorname{Arg}(z'-\alpha)^{1/2} = 0$ for z' real, $z' > \alpha$ and the branch $\operatorname{Arg}(z'-1)^{1/2} = 0$ for z' real, $z' > 1$. When (3.23) is transformed under this map we obtain,

$$\omega_1(z') = \frac{1}{\pi} \{(z'-1)(z'-\alpha)\}^{1/2} \int_1^{\alpha} \frac{\theta(u)}{\{(u-1)(\alpha-u)\}^{1/2}} \frac{du}{z'-u} . \qquad (5.3)$$

As a check it is simpler to derive (3.23) from (5.3) since,

$$\left\{ \frac{(z'-\alpha)(z'-1)}{(u-1)(\alpha-u)} \right\}^{1/2} = i \left\{ \frac{z^2-1}{1-x^2} \right\} ,$$

$$\frac{1}{z'-u} = \frac{(1+x^2)(1+z^2)}{(z-x)(1-xz)(\alpha-1)} ,$$

$$du = \frac{(\alpha-1)(1-x^2)}{(x^2+1)^2} dx .$$

Now substitute (5.1) into (5.3) and integrate with a change of variable,

$$t^2 = \frac{\alpha-u}{u-1} ,$$

to obtain,

$$\left\{\frac{z'-\alpha}{z'-1}\right\}^{1/2} \int_0^{t_0} \frac{dt}{t^2 + \left\{\frac{z'-\alpha}{z'-1}\right\}} \quad , \quad t_0^2 = \frac{\alpha-\alpha_0}{\alpha_0-1} \quad . \tag{5.4}$$

We shall choose the branches as specified above and the principal branch of the logarithm with the result that $\omega_1(z')$ is,

$$\frac{1}{2i} \operatorname{Log} \frac{\left\{\frac{z'-\alpha}{z'-1}\right\}^{1/2} + i\left\{\frac{\alpha-\alpha_0}{\alpha_0-1}\right\}^{1/2}}{\left\{\frac{z'-\alpha}{z'-1}\right\}^{1/2} - i\left\{\frac{\alpha-\alpha_0}{\alpha_0-1}\right\}^{1/2}} \quad . \tag{5.5}$$

A branch point occurs at $z' = \alpha_0$ but it turns out that for z' real and $1 \leqq z' = x' \leqq \alpha$,

$$\omega_1(x') = \frac{\pi}{2} - \frac{i}{2} \operatorname{Log} \left| \frac{\left\{\frac{\alpha-x'}{x'-1}\right\}^{1/2} + \left\{\frac{\alpha-\alpha_0}{\alpha_0-1}\right\}^{1/2}}{\left\{\frac{\alpha-x'}{x'-1}\right\}^{1/2} - \left\{\frac{\alpha-\alpha_0}{\alpha_0-1}\right\}^{1/2}} \right| \quad . \tag{5.6}$$

The boundaries of the flow are found from equation (3.21) when transformed to the z' plane. With $\beta = 1$ we obtain from (5.2),

$$dz' = (\alpha-1)\,\frac{(1-z^2)}{(1+z^2)^2}\, dz \;;$$

so expression (3.20) yields,

$$dz_0 = \frac{Q}{\pi V(A)} \, \frac{e^{i\omega(z')}}{\sqrt{z'}\,(z'-1)} \, dz' \quad .$$

Hence the boundaries of the flow for the n'th approximation are given by

$$z_o^n = \frac{Q}{\pi V(A)} \int_{x_1}^{x_o} \frac{e^{i\omega_n(x')}}{\sqrt{x'}\,(x'-1)}\, dx' \,. \tag{5.7}$$

The approximations ω_n for $n > 1$ are obtained from Ω_n for the upper unit semi-circle (3.25). Let s represent the arc length on the circumference of this semi-circle. We shall compute $\Omega_n(z')$ for the upper half z' plane by transforming (3.25) under the map (5.2). Define

$$\eta = \frac{\alpha+1}{2} + \frac{\alpha-1}{2\,\mathrm{Cos}\ s} \quad . \tag{5.11}$$

Then $\mathrm{Cos}\ s = \dfrac{\alpha+1}{2\eta-\alpha-1}$, $\mathrm{Tan}\ s = \dfrac{2(\eta-1)^{1/2}(\eta-\alpha)^{1/2}}{\alpha-1}$ where we have specified the branches above. Under the mapping (5.2) $R_1(s)ds$ of (3.25) becomes $R_2(\eta)d\eta$ where $R_2(\eta)$ is defined by,

$$R_2(\eta) = \begin{cases} 0 & \text{for} \quad \eta < 0 \,, \\ \dfrac{1}{\sqrt{\eta}\,(1-\eta)} & \text{for} \quad 0 \leqq \eta \leqq 1 \\ & \qquad\quad \alpha \leqq \eta < \infty \,, \end{cases}$$

and the logarithmic factor becomes $\mathrm{Log}\ N(\eta, z')$ where N is defined by the expression,

$$\frac{(\alpha-1)(z'-\eta)+i(2\eta-\alpha-1)(z'-1)^{\frac{1}{2}}(z'-\alpha)^{\frac{1}{2}}+i(2z'-\alpha-1)(\eta-1)^{\frac{1}{2}}(\eta-\alpha)^{\frac{1}{2}}}{(\alpha-1)(z'-\eta)+i(2\eta-\alpha-1)(z'-1)^{\frac{1}{2}}(z'-\alpha)^{\frac{1}{2}}-i(2z'-\alpha-1)(\eta-1)^{\frac{1}{2}}(\eta-\alpha)^{\frac{1}{2}}} \,. \tag{5.12}$$

By our convention adopted above $|N| \nleq 1$ which agrees with the magnitude of argument of the logarithm as it occurs in expressions (3.25) and (3.27). For example $(z'-1)^{1/2}\ (z'-\alpha)^{1/2}$ is positive for $z' > \alpha$ and negative

for $z' < 1$. The same is true for η so $\mathrm{Log}\ |N| > 0$.

The expression Ω_n is thus

$$\Omega_n(z') = \frac{\mu}{\pi} \int_D \sin f_{n-1}(\eta) e^{-3g_{n-1}(\eta)} \mathrm{Log}\, \frac{N(\eta,z')}{N(\eta,1)}\, B(\eta) d\eta \,, \qquad (5.13)$$

where

$$D = [0,1] + [\alpha,\infty),$$

and

$$B(\eta) = \frac{1}{\sqrt{\eta}\,(1-\eta)} \quad . \qquad (5.14)$$

If we compare this expression with the corresponding expression obtained in reference [8] we find that $B(\eta)$ must be defined to be

$$F(\eta) = \frac{(\alpha-1)}{(1-\eta)(2\eta-\alpha-1)\eta^{1/2}(\eta-1)^{1/2}(\eta-\alpha)^{1/2}} \quad . \qquad (5.15)$$

We shall try to find some indication of how the $B(\eta)$ as defined in expressions (5.14) and (5.15) affect the flow. We shall attempt several calculations by hand to illustrate the procedure in evaluating the above integrals and to try to achieve some insight into their behavior and the feasibility of using an automatic computor.

The quantity a which is the potential difference between the edge of the weir and the equi-potential CC' is chosen to be 2Q . From the expression (3.3) α is found to be 1.0075 . The values of the parameters H and V(A) are one meter and ten meters per second,

so that Q is ten cubic meters per second, the Froude number F_r is about ten and the parameter μ is about .03 .

The height of the weir is found in conjunction with expression (5.6) and (5.7) to be,

$$h = \mathrm{Im}\left\{ \frac{Q}{\pi V(A)} \int_{\alpha_o}^{\alpha} \left\{ \frac{\{(x'-1)(\alpha-\alpha_o)\}^{1/2}+\{(\alpha-x')(\alpha_o-1)\}^{1/2}}{\{(x'-1)(\alpha-\alpha_o)\}^{1/2}-\{(\alpha-x')(\alpha_o-1)\}^{1/2}} \right\}^{1/2} \frac{dx'}{\sqrt{x'}\,(x'-1)} \right. \circ$$

Here α_o appears in the integrand as well as in the limits of integration and h is known so it is difficult to determine α_o from h. One must substitute a sequence of α_o and compute the corresponding sequence of h and then choose the value of α_o which closest approximates h. In this example α_o is chosen to be 1.0025 and the height is computed using the method of tangents and the above parameters. The computation is contained in Table 1. The height is found to be about .831 meters which is less than the height H of the flow.

The remaining boundaries of the flow are found from (5.7) using the expression for $\omega_1(x')$ as derived in expression (5.5). The result is,

$$z_o^1 = \frac{1}{\pi} \int_{x_1'}^{x_o'} P(x')dx', \qquad (5.8)$$

where $P(x')$ is the integrand of (5.7) namely,

$$P(x') = \left\{ \frac{(\alpha_o-1)(x'-\alpha)}{(\alpha-1)(x'-\alpha_o)x'} \right\}^{1/2} \left\{ 1 + i \left\{ \frac{(\alpha-\alpha_o)(x'-1)}{(\alpha_o-1)(x'-\alpha)} \right\}^{1/2} \right\} \frac{1}{x'-1} \cdot \qquad (5.9)$$

The limits of integration x_1' and x_o' must correspond to the stream boundary as represented in the z' plane. For example, if we wish to find the lower free streamline we set $x_1' = \alpha$ and vary x_o' along the real axis $x' > \alpha$. The expression (5.8) is evaluated by the method of tangents. In table two the numerical computations are recorded. The lower free streamline is found to be higher than that shown in in figure 6 and it becomes straight after one half the distance between B and C. For further information about the boundaries of the first approximation one should refer to reference [8]. Appendix 2 contains a brief discussion on the method of tangents.

Table 1

Height of Weir

1	2	3	4	5	6
x	$\{\sqrt{x}\,(x-1)\}^{-1}$	A	dx	dI	h
1.003	332.384	3.731	.0001	1.242	-
1.004	249.501	2.215	.0001	.553	-
1.005	199.501	1.732	.0001	.346	-
1.006	166.169	1.447	.0001	.241	-
1.007	142.359	1.211	.0001	.172	.813

$$A = \left\{ \frac{\sqrt{2}(x-1)^{1/2} + (\alpha-x)^{1/2}}{\sqrt{2}\,(x-1)^{1/2} - (\alpha-x)^{1/2}} \right\}^{1/2}, \qquad h = \frac{1}{\pi} \Sigma\, dI .$$

Since the first approximation in reference [8] agrees with those obtained here some of the numerical results for this approximation from reference [8] are entered in tables one and two.

Table 2

Lower Free Streamline

x	dx	$\left(\frac{2x-2}{x-\alpha}\right)^{1/2}$	$\frac{1}{\sqrt{x}\,(x-1)}$	dx_1	dy_1	x_1	y_1
1.01	.005	2.828	99.50	.053	.149	.050	.965
1.019	.0125	1.812	52.14	.099	.182	.153	1.144
1.0375	.025	1.581	26.18	.116	.184	.269	1.328
1.0625	.025	1.503	15.52	.067	.103	.337	1.431
1.0875	.025	1.474	10.96	.049	.072	.385	1.503
1.112	.025	1.47	8.47	.038	.056	.424	1.559
1.2	.15	1.44	4.57	.124	.178	.548	1.737
1.4	.25	1.43	2.12	.096	.138	.645	1.875
1.76	.47	1.42	.99	.086	.123	.731	1.998
3.	2.	1.41	.29	.107	.149	.838	2.147
5.	2.	1.41	.11	.042	.058	.879	2.205
8.	4.	1.41	.05	.037	.052	.916	2.257
15.	10.	1.41	.02	.034	.048	.950	2.305
40.	40.	1.41	.004	.030	.042	.980	2.347
100.	80.	1.41	.001	.015	.021	.994	2.368
240.	200.	1.41	.0002	.010	.0123	1.005	2.380

When z' is real and is not contained in the interval $[1,\alpha]$ we can write $\omega_1(x')$ in the form,

$$\operatorname{Arctan}\left\{\frac{(\alpha-\alpha_o)(x'-1)}{(\alpha_o-1)(x'-\alpha)}\right\}^{1/2}, \tag{5.10}$$

which is real if x' satisfies these conditions. This expression is obtained from equation (5.5). We can thus write

$$g_1(x',0) = 0 ,$$

$$\sin f_1(x',0) = \left\{ \frac{(\alpha-\alpha_o)(x'-1)}{(\alpha-1)(x'-\alpha_o)} \right\}^{1/2} .$$

When these expressions are substituted into equation (5.13) the result is,

$$\Omega_2(z') = \frac{\mu}{\pi} \int_D \left\{ \frac{(\alpha-\alpha_o)(\eta-1)}{\eta(\alpha-1)(\eta-\alpha_o)} \right\}^{1/2} \operatorname{Log} \frac{N(\eta,z')}{N(\eta,1)} \; \frac{d\eta}{1-\eta} . \qquad (5.11)$$

The boundaries of the flow for the second approximation are found from expression (5.7),

$$z_o^2 = \frac{Q}{\pi V(A)} \int_{x_1}^{x_o} e^{iF_2(x')} \; e^{-G_2(x')} \left\{ \frac{e^{i\omega_1(x')}}{\sqrt{x'}\,(x'-1)} \; dx' \right\} . \qquad (5.12)$$

The quantity in the brackets in this expression has been computed by the method of tangents for a set of points since it does represent the flow boundaries for the first approximation. Consequently if we evaluate z_o^2 at the same set of points then the vector $dx_1 + idy_1$ in table 2 representing the increments in the flow boundaries for the first approximation is essentially multiplied by the vector $\exp i\,\Omega_2$. The effect of this operation is to cause a rotation of the increment vector $dx_1 + idy_1$ and an extension. On examination we find the vector is rotated an angle $F_2(x')$ so we can obtain some insight into what is going on by examining the behavior for $F_2(x')$. One would expect in the case of the free streamlines that $F_2(x')$ the angle of rotation be negative because gravity acts in such a way as to pull down the free streamlines (L) and (L_1) from the first approximation where gravity is zero.

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On examining (5.11) we find that $\sin f_1$ is always positive in the interval $[0,1] + [\alpha,\infty)$. Also by our convention the logarithmic factor is always positive. Hence $F_2(x')$ consists of a positive contribution arising from the interval $[0,1]$ and a negative contribution arising from the interval $[\alpha,\infty)$ when x' corresponds to the interval $[\alpha,\infty)$. For the lower free streamline we would expect that $F_2(x') < 0$.

If we compare these results with those of reference [8] we find that in using $B(\eta)$ of (5.15) instead of (5.14), the effect is to scale down the logarithmic factor in Ω_2 for $\eta > \alpha$ and increase it for $\eta < 1$. In fact for large η, (5.14) behaves like $\eta^{-3/2}$ while (5.15) behaves like $\eta^{-7/2}(\alpha-1)$. Consequently the contribution from the values $\eta \geqq 10$ are significant for (5.14) while for (5.15) they are not. So more computations are necessary than those done by reference [8].

This discussion means that most of the contribution to $F_2(x')$ for $x' > \alpha$ is coming from $[0,1]$ using (5.15) while in the case of (5.14) it is the interval $[\alpha,\infty)$. Hence the results will be such that $F_2(x')$ will be negative in one case and positive in the other.

I have computed $F_2(x')$ for various values of x'. Table 3 shows the results for $F_2(15)$. In general when x' corresponds to the upper free or lower free streamline then F_2 results from a negative and a positive contribution as exhibited in table 3. These two contributions are almost equal is magnitude and hence the sum is very small. Consequently a great number of points should be taken because accuracy is very important.

However I did find that F_2 was negative on the interval $[\alpha,\infty)$ and positive on the interval $[0,1]$. When x' corresponds to the lower free streamline then the streamline is lowering which agree with the intuitive argument presented above. The upper free streamline should be changed also depending upon a number of factors but we have found that for our computation the ascent is steeper for the second approximation than for the first. If one had a large computor these computations could be done more accurately and more quickly and the whole flow could be analysed. But this is not our intensions here. We simply want to investigate this method as a means of analysing gravity flows. For more numerical results see Appendix 3.

Table 3

$X = 15.0$

η	$\Delta\eta$	Log N	$\sqrt{(\eta)(\eta-1)(\eta-\alpha_o)}$	$\frac{\text{Log N}}{\sqrt{\eta(\eta-1)(\eta-\alpha_o)}}$	dI
.02	.03	6.19	.139	44.57	1.34
.05	.04	6.17	.213	29.36	1.17
.1	.10	6.11	.285	17.91	1.79
.2	.13	6.01	.358	16.76	2.18
.4	.2	5.73	.380	15.07	3.02
.6	.2	5.34	.311	17.19	3.44
.8	.2	4.67	.180	25.96	5.19
.95	.09	3.35	.050	67.02	6.03
.995	.01	1.49	.006	243.34	2.43
1.01	.005	1.10	- .0087	-126.32	- .63
1.05	.0675	3.33	- .050	- 66.48	-5.82
1.1	.04	3.87	- .101	38.32	-1.52
1.55	.88	5.71	- .683	- 8.36	-7.36
3.0	2.0	7.13	- 3.462	- 2.06	-4.12
6.0	4.0	9.48	-12.24	- .774	-3.09
11.0	4.5	9.83	-33.17	- .300	-1.33
14.0	2.5	11.48	-48.64	- .246	- .59
16.0	3.0	11.63	-60.00	- .194	- .59
20.0	17.0	10.25	-84.97	- .121	-2.06
50.0	40.0	9.25	-346.5	- .027	-1.07
100.0	100.0	9.07	-1000.	- .009	- .41
1000.	5×10^3	8.93	-31620.	-2.8×10^{-4}	-1.27
10000.	5×10^4	8.92	$-1. \times 10^6$	-8.9×10^{-6}	- .45

$\Sigma\, dI = -3.7$

$F_2(15) = \frac{\mu}{\pi}\Sigma\, dI = -.037$ radians .

Example 2

We have seen in the previous example that it was difficult to determine α_o for a given value of the weir height because the height was determined from an integral equation where α_o appeared in the limits of integration as well as in the integrand. In general if any constants occur in the definition of $\theta(x')$ in the interval $[\beta,\alpha]$ of the complex z' plane or in the interval $[-1,+1]$ of the complex z plane then some difficulty of this type will occur. The reason for this is that $\omega_1(z')$ in this interval will be a function of these constants. The constants are determined by the shape of the surface (S) which is known. We determine the shape of (S) by the expression z_o^1 for given constants. In this expression the constants will appear in the integrand and in the limits of integration. Usually it is impossible to derive an expression for these constants in terms of known parameters in $\theta(x')$ so some numerical method must be used as we have found out in the previous example.

As a further illustration of the problems to which this theory is applicable and as an illustration of how one should interpret the theoretical results we shall outline the approach to another problem which has interested many people.

Let us consider the flow of a fluid through a semi-infinite pipe which is partially closed at one end. See figure 7.

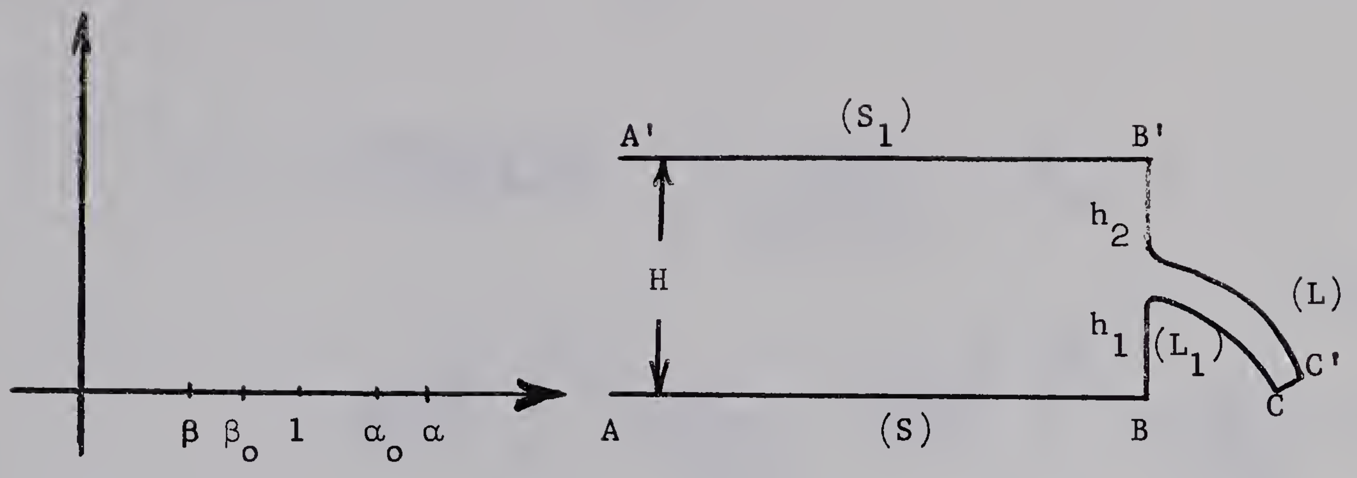

Figure 7

We shall refer to the upper half z' plane instead of the unit semi-circle because of certain symmetry conditions which will be observed as this discussion proceeds. Let us define $\theta(x)$ to be,

$$\theta(x) = \begin{cases} -\pi/2 & \text{for} \quad \beta \leqq x < \beta_o \\ 0 & \text{for} \quad \beta \leqq x \leqq \alpha_o \\ +\pi/2 & \text{for} \quad \alpha_o < x \leqq \alpha . \end{cases} \tag{5.21}$$

The interval $[\beta,1]$ will refer to the surface (S_1) while the interval $[1,\alpha]$ will refer to the surface (S) .

We shall use the same numerical values for the parameters of the flow as was used in the previous example. Hence the velocity $V(A)$ is ten meters per second and the height H is one meter so that Froude number Fr is about 10 and the parameter μ is about .031.

To obtain $\omega_1(z')$ we must transform expression (3.23) under the mapping (3.4). With θ defined above we obtain,

$$\omega_1(z') = \frac{\sqrt{(z'-\alpha)(z'-\beta)}}{\pi} \int_\beta^\alpha \frac{\theta(u)}{\sqrt{(u-\beta)(\alpha-u)}} \frac{du}{z'-u} + iC\,,$$

$$= \left\{\frac{z'-\alpha}{z'-\beta}\right\}^{\frac{1}{2}} \int_0^{\xi_1} \frac{dV}{V^2+\left\{\frac{z'-\alpha}{z'-\beta}\right\}} - \left\{\frac{z'-\beta}{z'-\alpha}\right\}^{\frac{1}{2}} \int_0^{\xi_2} \frac{dV}{V^2+\left\{\frac{z'-\beta}{z'-\alpha}\right\}} + iC\,, \qquad (5.22)$$

for

$$\xi_1 = \left\{\frac{\alpha-\alpha_o}{\alpha_o-\beta}\right\}^{1/2}, \quad \text{and} \quad \xi_2 = \left\{\frac{\beta_o-\beta}{\alpha-\beta_o}\right\}^{1/2};$$

where C is a constant determined by the boundary condition $\omega_1(1) = 0$. We shall choose the branches $\operatorname{Arg}(z'-\alpha)^{1/2}$ is zero for z' real and greater than α and $\operatorname{Arg}(z'-\beta)^{1/2}$ is zero for z' real and greater than one. On taking the principal value of the logarithm we obtain the expression for $\omega_1(z')$,

$$\frac{1}{2i}\left\{ \operatorname{Log} \frac{\left\{\frac{z'-\alpha}{z'-\beta}\right\}^{\frac{1}{2}} + i\left\{\frac{\alpha-\alpha_o}{\alpha_o-\beta}\right\}^{\frac{1}{2}}}{\left\{\frac{z'-\alpha}{z'-\beta}\right\}^{\frac{1}{2}} - i\left\{\frac{\alpha-\alpha_o}{\alpha_o-\beta}\right\}^{\frac{1}{2}}} - \operatorname{Log} \frac{\left\{\frac{z'-\beta}{z'-\alpha}\right\}^{\frac{1}{2}} + i\left\{\frac{\beta_o-\beta}{\alpha-\beta_o}\right\}^{\frac{1}{2}}}{\left\{\frac{z'-\beta}{z'-\alpha}\right\}^{\frac{1}{2}} - i\left\{\frac{\beta_o-\beta}{\alpha-\beta_o}\right\}^{\frac{1}{2}}} \right\} + iC\,.$$

The point $z' = \alpha_o$ is a branch point of the first term and the point $z' = \beta_o$ is a branch point of the second term. Consequently one must be careful in interpreting this function on the real axis.

It turns out that four cases must be considered. The appropriate results are,

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[illegible]

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[illegible]

[illegible]

[illegible]

[illegible]

Case 1. $\qquad x' > \alpha \quad$ or $\quad 0 < x' < \beta$, $\qquad$ (5.24)

$$\omega_1(x') = \text{Arctan}\left\{\frac{(\alpha-\alpha_o)(x'-\beta)}{(\alpha_o-\beta)(x'-\alpha)}\right\}^{\frac{1}{2}} - \text{Arctan}\left\{\frac{(\beta_o-\beta)(x'-\alpha)}{(\alpha-\beta_o)(x'-\beta)}\right\}^{\frac{1}{2}} + iC,$$

Case 2. $\qquad \beta_o < x' < \alpha_o$, $\qquad$ (5.25)

$$\omega_1(x') = \frac{1}{2i}\,\text{Log}\;|X(x') \times Y(x')| + iC,$$

Case 3. $\qquad \alpha_o < x' < \alpha$, $\qquad$ (5.26)

$$\omega_1(x') = \frac{\pi}{2} + \frac{1}{2i}\,\text{Log}\;|X(x') \times Y(x')| + iC,$$

Case 4. $\qquad \beta < x' < \beta_o$, $\qquad$ (5.27)

$$\omega_1(x') = -\frac{\pi}{2} + \frac{1}{2i}\text{Log}\;|X(x') \times Y(x')| + iC,$$

where

$$X(x') = \frac{\left\{\frac{\alpha-x'}{x'-\beta}\right\}^{\frac{1}{2}} + \left\{\frac{\alpha-\alpha_o}{\alpha_o-\beta}\right\}^{\frac{1}{2}}}{\left\{\frac{\alpha-x'}{x'-\beta}\right\}^{\frac{1}{2}} - \left\{\frac{\alpha-\alpha_o}{\alpha_o-\beta}\right\}^{\frac{1}{2}}}, \qquad Y(x') = \frac{\left\{\frac{x'-\beta}{\alpha-x'}\right\}^{\frac{1}{2}} + \left\{\frac{\beta_o-\beta}{\alpha-\beta_o}\right\}^{\frac{1}{2}}}{\left\{\frac{x'-\beta}{\alpha-x'}\right\}^{\frac{1}{2}} - \left\{\frac{\beta_o-\beta}{\alpha-\beta_o}\right\}^{\frac{1}{2}}}; \qquad (5.28)$$

and where C is determined from the boundary condition $\omega_1(1) = 0$.

From case 2 then,

$$C = \frac{1}{2} \text{Log} \; |X(1) \times Y(1)| \quad . \tag{5.29}$$

As in the previous example we shall choose $a = 2Q$. Due to the conditions of symmetry in the surfaces (S) and (S_1) it turns out that there is also symmetry in the potentials of the flow under no gravity. Hence we set $b = a$. α is determined from expression (3.3) to be 1.0075 and $\beta = 1/\alpha$. Due to the symmetry too we know that the potentials at the points B and B' are equal. Then we set $\alpha_o = 1.0025$ and β_o is determined to be $1/\alpha_o$ from (3.3). We must now check to see if the heights h_1 and h_2 as indicated in figure 7 total a number less than one.

Without the symmetry that we have assumed here the constants α , α_o, β and β_o must be altered in such a way that the computed boundaries coincide with the given ones. Sometimes this is difficult to achieve because one does not know in general how any variation in any combination of these constants will affect the boundary of the flow. Under these symmetry conditions we have,

$$x(1/x') = Y(x'), \tag{5.30}$$

so

$$C = \text{Log} \; |X(1)| = \text{Log} \; 5.813 \quad .$$

The formula for h_1 is found using ω_1 as derived in case 3.

$$h_1 = I_m \left\{ \frac{1}{\pi} \int_{\alpha_o}^{\alpha} \frac{e^{i\omega_1(x')}}{\sqrt{x'}\,(x'-1)}\, dx' \right\};$$

$$= \frac{1}{5.813\,\pi} \int_{\alpha_o}^{\alpha} \left\{ \frac{|X(x') \times Y(x')|}{x'} \right\}^{\frac{1}{2}} \frac{dx'}{x'-1} . \tag{5.31}$$

Under the substitution $x' = 1/y'$ in this equation we obtain an equation which is idential with the expression for $-h_2$. This means that $h_1 = -h_2$ as one should expect. Using the method of tangents the height h_1 can be computed. The computation is analogous to that shown in table 1 of the first example. We use the same set of points also. The height is found to be about .3 meters which is clearly less than half the height H of the pipe.

We shall observe more symmetry in the boundary of the flow for the first approximation. The lower free streamline (L_1) is found by using ω_1 of case 1 above. Hence the expression is,

$$z_o^1 = \frac{1}{\pi} \int_{\alpha}^{x' \to \infty} \frac{e^{i\omega_1(x')}}{\sqrt{x'}\,(x'-1)}\, dx' . \tag{5.32}$$

Under the change of variable $x' = 1/y'$ we obtain,

$$z_o^1 = \frac{1}{\pi} \int_{\beta}^{y' \to o} \frac{e^{i\omega_1(1/y')}}{\sqrt{y'}\,(y'-1)}\, dy',$$

$$= \frac{1}{\pi} \int_{\beta}^{y' \to o} \frac{e^{-i\omega_1(y')}}{\sqrt{y'}\,(y'-1)}\, dy', \tag{5.33}$$

which is the expression for the upper free streamline (L), providing the exponent in the integrand be changed to $+ i\omega_1(y')$.

This means that the upper and lower free streamlines as well as the surfaces (S) and (S_1) are symmetric with respect to a line half way between the surfaces (S_1) and (S) and parallel to them. The real part of the function ω_1 represents the slope of the free streamlines of $x' > \alpha$ or $0 < x' < \beta$. In table four is recorded this slope for various values of η .

Table 4

η	$f_1(\eta)$	η	$f_1(\eta)$
.05	+ .003	20.0	- .003
.10	+ .004	10.0	- .004
.20	+ .005	5.0	- .005
.50	+ .011	2.0	- .011
.67	+ .018	1.5	- .018
.91	+ .075	1.1	- .075
.95	+ .181	1.05	- .181
.99	+ .823	1.01	- .823

As we can see the slope becomes zero very quickly which means the streamlines become horizontal very quickly.

The approximations of order greater than one are obtained in a manner previously outlined. $\Omega_2(z')$ is given by an expression similar to (5.13). Hence $\Omega_2(z')$ is,

$$\frac{\mu}{\pi} \int_D \sin f_1(\eta)\, e^{-3g_1(\eta)} \operatorname{Log} \frac{N(\eta,z')}{N(\eta,1)} \frac{d\eta}{\sqrt{\eta}\,(1-\eta)} , \tag{5.34}$$

where $D = [0,\infty) - [\alpha,\beta]$, $f_1(\eta)$ and $g_1(\eta)$ are obtained from case 1 above and $N(\eta,z')$ is defined by,

$$\frac{(\alpha-\beta)(z'-\eta)+(2\eta-\alpha-\beta)\{(z'-\beta)(\alpha-z')\}^{\frac{1}{2}}+i(2z'-\alpha-\beta)\{(\eta-\alpha)(\eta-\beta)\}^{\frac{1}{2}}}{(\alpha-\beta)(z'-\eta)+(2\eta-\alpha-\beta)\{(z'-\beta)(\alpha-z')\}^{\frac{1}{2}}-i(2z'-\alpha-\beta)\{(\eta-\alpha)(\eta-\beta)\}^{\frac{1}{2}}} . \tag{5.35}$$

This must be interpreted as in example one. Hence $|N(\eta,z')| \geqq 1$.

The lower free streamline for the second approximation is given by,

$$z_o^2 = \frac{1}{\pi} \int_\alpha^\infty \frac{e^{i\omega_1(x')}}{\sqrt{x'}\,(x'-1)} \times e^{i\Omega_2(x')}\, dx', \tag{5.36}$$

where $\omega_1(x')$ is obtained from case 1 above and $\Omega_2(x')$ is obtained from (5.34).

Now the real part of $\Omega_2(x')$ will represent the change in the slope of the lower free streamline for the first approximation. In the interval $[0,\beta]$ $\sin f_1$ is negative while in the interval $[\alpha,\infty)$ it is positive. Consequently the real part of $\Omega_2(x')$ for $x' > \alpha$ will be negative.

This is in agreement with our intuitive argument in which gravity acts in such a way as to lower the lower free streamline from the first approximation.

In table 5 is recorded a typical hand computation of $F_2(x')$ for $x' > \alpha$. This is intended only as an indication of the procedure

in evaluating the flow boundaries. One finds the greatest contribution of the integral comes from the interval $[.4,\beta] \cup [\alpha,2]$ since outside of this $\sin f_1$ is almost zero. No attempt will be made to continue computation on the second approximation or to proceed to the third appoximation without the use of a computor which clearly is necessary.

Table 5

$x' = 15.0$

1 η	2 $\Delta\eta$	3 $\text{Log}\,\lvert N(\eta,x')\rvert$	4 $\text{Sin}\ f_1(\eta)$	5 $\sqrt{\eta}\ (1-\eta)$	6 dI
.02	.04	5.502	- .002	.139	- .003
.05	.02	5.473	- .003	.212	- .002
.07	.02	5.453	- .003	.246	- .002
.10	.04	5.422	- .004	.285	- .003
.20	.16	5.311	- .005	.358	- .012
.40	.22	5.037	- .008	.380	- .023
.60	.20	4.644	- .014	.310	- .042
.80	.15	3.965	- .032	.179	- .106
.90	.08	3.280	- .067	.095	- .185
.95	.03	2.591	- .137	.049	- .217
.97	.02	2.072	- .187	.030	- .262
.99	.02	.802	- .723	.010	-1.161
1.01	.005	.800	.723	- .010	- .292
1.03	.02	2.070	.24	- .030	- .320
1.05	.0425	2.595	.140	- .051	- .302
1.10	.125	3.295	.074	- .105	- .291
1.30	.20	4.407	.026	- .342	- .067
1.50	.30	4.932	.018	- .614	- .044
2.00	.80	5.664	.011	-1.414	- .035
3.00	1.00	6.051	.007	-3.460	- .012
4.00	1.00	6.929	.006	-6.000	- .007
6.00	3.00	7.641	.005	-12.25	- .009
10.00	5.00	8.817	.004	-28.85	- .006
14.00	2.50	11.51	.003	-48.64	- .002
16.00	2.50	11.60	.003	-60.00	- .001
25.00	15.00	7.54	.002	-120.00	- .002

$\Sigma\ dI = -3.408$

$$dI = \Delta\eta\ \text{Log}\ \lvert N(\eta,15)\rvert \cdot \frac{\sin f_1(\eta)}{\sqrt{\eta}\ (1-\eta)}\ , \qquad F_2(15) = \frac{\mu}{\pi}\, e^{-3g_1}\ \Sigma\ dI = -\ .00002.$$

For more results see Appendix 3.

CHAPTER VI

CONCLUSIONS

Under the assumption imposed upon the flow in chapter three we have investigated the flow under gravity of a fluid flowing over a bed (S) and falling freely for some distance. We have established the uniform convergence of the sequence of the successive approximations to a function which satisfies the boundary conditions. We have outlined the procedure of applying these results to concrete flow falls by reference to two examples.

We have assumed explicitly that an equipotential line occurs at a point downstream of the point of overflow, which is such that the variation of the magnitude of the velocity along it is zero. In actual fact this occurs only an infinite distance downstream but it has been necessary to terminate the downstream flow domain by such an equipotential in order to eliminate a theoretical difficulty which otherwise occurs. This assumption may be satisfied along any equipotential downstream by placing sources and sinks at various positions outside the downstream flow domain but this affects the whole flow and the origional problem is changed. The effect of such an assumption is such that the nearer the equipotential line is to the point of overflow the more serious are the deviations from the origional problem. Far downstream, the velocity becomes essentially vertical, so from the Bernoulli equation $V^2 + 2gy = \text{constant}$, it is evident that the deviation in the magnitude of the velocity along every equipotential line is

very small and the effect of the above assumption is negligible.

In the numerical examples that have been examined above, the deviations between the first and second approximations to the flow, as are evident from tables 3 and 4, are found to be very small. This could be attributed to the fact that the equipotential line CC' was too close to the point of overflow. If this equipotential line is moved a greater distance downstream, then the deviation of the velocity along this equipotential is negligible and can be assumed to be zero. This will make some difference in the numerical results and the above discussion points to the fact that the results could be more favourable. Due to the small size of the results we have obtained, and the existence of singularities in the integrand of the solutions, accuracy will necessitate the use of a large automatic computor. A computor study should be carried out before definite conclusion can be drawn.

The pressure can be determined to within a constant at any point of the fluid only after the problem is completely solved. This constant cannot be determined because the origional pressure p, as it occurred in the Bernoulli equation, was constant along the free streamlines and does not appear when the Bernoulli equation is differentiated. In assuming that an equipotential occurs at a finite distance downstream, along which the variation of the velocity is zero, we have, it appears, assumed that outside the flow domain there occur sources or sinks which make this possible. It is not possible, when the problem is solved, to determine what happens outside this flow domain.

APPENDIX I

Continuity of $F_n(1,s)$ and $G_n(1,s)$

Define $\Phi_n(u) = \sin f_{n-1}(1,u)\, e^{-3g_{n-1}(1,u)} R_1(u)du$ so that

$$F_n(1,s) = \frac{\mu}{\pi} \int_0^{\pi} \Phi_n(u) \operatorname{Log} \left| \frac{\sin (u+s)/2}{\sin (u-s)/2} \right| du \quad .$$

Since $R_1(u)$ is zero for $\gamma > u > \pi/2$ the integral in $F_n(1,s)$ can be written as the sum of two integrals whose limits of integration extend from 0 to $\pi/2$ and from γ to π. These two integrals are very similar since the type of singularity and order of the singularity are the same. We shall limit ourselves to the former of these two integrals since continuity of the latter is analogous to that of the former. But we should first make a few observations.

Note 1. Log sin u is Lipschitz continuous for u in the interval $[a,b]$ where $0 < a < b \leqq \pi/2$.

Note 2. For $s \leqq \pi/2$

$$\left| \int_0^s \operatorname{Log} \sin u \, du \right| = \left| \int_0^s (\operatorname{Log} u + \operatorname{Log} \frac{\sin u}{u} du \right| ,$$

$$\leqq \left| (\operatorname{Log} s - 1)s \right| + \left| \int_0^s \operatorname{Log} \frac{\sin u}{u} du \right| ,$$

$$\leqq |s| \left\{ |\operatorname{Log} s - 1| + \left|\operatorname{Log} \frac{\sin A}{A}\right| \right\} ,$$

$$= |s| \cdot N \leqq \frac{\pi}{2} N ,$$

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for some $N > 0$ and where $\text{Log}\,\frac{\sin A}{A}$ is the maximum value of $\text{Log}\,\frac{\sin u}{u}$ for $0 \leqq u \leqq \pi/2$.

Note 3. The integral above exists since all of the singularities are of order less than one.

Note 4. $\Phi_n(u)$ is continuous in the interval $[0,d]$ where $d < \pi/2$ and where we shall assume that the approximations $f_{n-1}(1,u)$ and $g_{n-1}(1,u)$ are bounded and continuous on the circumference of the unit semi-circle.

The following cases must be examined in order to prove that $F_n(1,s)$ is continuous.

Case 1. $0 < s < s' < \pi/2$,

Case 2. $\pi/2 < s < s' < \pi$,

Case 3. $s \to \pi/2^-$,

Case 4. $s \to \pi/2^+$,

Case 5. $s \to 0$,

Case 6. $s \to \pi$,

We shall illustrate the procedure by considering only case one. In this case,

$$0 < \frac{u+s}{2} < \frac{\pi}{2} \quad \text{and} \quad 0 < \frac{u+s'}{2} < \frac{\pi}{2}, \quad \text{so}$$

$$|F_n(1,s)-F_n(1,s')| = \frac{\mu}{\pi}\left|\int_0^{\frac{\pi}{2}} \Phi_n(u)\left\{\text{Log}\left|\frac{\sin(u+s)/2}{\sin(u-s)/2}\right| - \text{Log}\left|\frac{\sin(u+s')/2}{\sin(u-s')/2}\right|\right\} du\right| .$$

a) $$\left|\int_0^{\frac{\pi}{2}} \Phi_n(u)\{\text{Log}\sin\frac{u+s}{2} - \text{Log}\sin\frac{u+s'}{2}\} du\right| ,$$

$$\leq \int_0^{\frac{\pi}{2}} |\Phi_n(u)|du \times M|s-s'| < \epsilon ,$$

providing $|s-s'| < \delta_1$ since $|\Phi_n(u)|$ is integrable and the statement of note one.

b) $$\left|\int_0^{\frac{\pi}{2}} \Phi_n(u)\{\text{Log}\sin\left|\frac{u-s}{2}\right| - \text{Log}\sin\left|\frac{u-s'}{2}\right|\} du\right| ,$$

$$= \left|\int_0^{s} \Phi_n(u)\ \{\text{Log}\sin\frac{s-u}{2} - \text{Log}\sin\frac{s'-u}{2}\} du\right.$$

$$+ \int_s^{s'} \Phi_n(u)\ \{\text{Log}\sin\frac{u-s}{2} - \text{Log}\sin\frac{s'-u}{2}\} du$$

$$\left.+ \int_{s'}^{\frac{\pi}{2}} \Phi_n(u)\ \{\text{Log}\sin\frac{u-s}{2} - \text{Log}\sin\frac{u-s'}{2}\} du\right| ,$$

$$= \left|2\int_0^{\frac{s}{2}} [\Phi_n(s-2v) - \Phi_n(s'-2v)]\ \text{Log}\sin v dv\right.$$

$$+ 2\int_0^{\frac{\pi}{8}-\frac{s'}{4}} [\Phi_n(s+2v)-\Phi_n(s'+2v)]\ \text{Log}\sin v dv$$

$$- 2\int_{\frac{s}{2}}^{\frac{s'}{2}} \Phi_n(s'-2v)\ \text{Log}\sin v dv$$

$$- 2\int_{\frac{\pi}{8}-\frac{s'}{4}}^{\frac{\pi}{4}-\frac{s'}{2}} \Phi_n(s'+2v)\ \text{Log}\sin v dv ,$$

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$$+ 2 \int_{\frac{\pi}{8} - \frac{s'}{4}}^{\frac{\pi}{4} - \frac{s}{2}} \Phi_n(s+2v) \text{ Log } \sin v dv| \quad .$$

Now we shall deal with each of these integrals separately.

c) $$|\int_0^{\frac{s}{2}} \{\Phi_n(s-2v)-\Phi_n(s'-2v)\} \text{ Log } \sin v dv,$$

$$\leq C_1 \ \epsilon_1 \int_0^{\frac{s}{2}} |\text{Log } \sin v| \ dv < \epsilon ,$$

providing $|s-s'| < \delta_2$ and by notes two and four with constant $C_1 > 0$ and $\epsilon > 0$ given.

d) $$|\int_0^{\frac{\pi}{8} - \frac{s'}{4}} \{\Phi_n(s+2v)-\Phi_n(s'+2v)\} \text{ Log } \sin v dv| ,$$

$$\leq \int_0^{\frac{\pi}{8} - \frac{s'}{4}} |\Phi_n(s+2v)-\Phi_n(s'+2v)| \times |\text{Log } \sin v| \ dv ,$$

$$\leq c_2 \ \epsilon_2 \int_0^{\frac{\pi}{8} - \frac{s'}{4}} |\text{Log } \sin v| \ dv < \epsilon ,$$

providing $|s-s'| < \delta_3$ and by notes two and four with constant $c_2 > 0$ and $\epsilon > 0$ given.

e) $$|\int_{\frac{s}{2}}^{\frac{s'}{2}} \Phi_n(s'-2v) \text{ Log } \sin v dv| ,$$

$$\leq \int_{\frac{s}{2}}^{\frac{s'}{2}} |\Phi_n(s'-2v) \text{ Log } \sin v| \ dv \leq R|s-s'|/2 < \epsilon ,$$

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providing $|s-s'| < \delta_4$ for $\epsilon > 0$ given and using the fact that the integrand is bounded in this interval.

f) $$\Big| -2 \int_{\frac{\pi}{8} - \frac{s'}{4}}^{\frac{\pi}{4} - \frac{s'}{2}} \Phi_n(s'+2v) \text{ Log sin } v dv$$

$$+ 2 \int_{\frac{\pi}{8} - \frac{s'}{4}}^{\frac{\pi}{4} - \frac{s}{2}} \Phi_n(s+2v) \text{ Log sin } v dv \Big| ,$$

$$= \Big| - \int_{\frac{\pi}{4} + \frac{s'}{2}}^{\frac{\pi}{2}} \Phi_n(v) \text{ Log sin } \frac{v-s'}{2} dv$$

$$+ \int_{\frac{\pi}{4} - \frac{s'}{2} + s}^{\frac{\pi}{2}} \Phi_n(v) \text{ Log sin } \frac{v-s}{2} dv \Big| ,$$

$$= \Big| \int_{\frac{\pi}{4} + \frac{s'}{2}}^{\frac{\pi}{2}} \Phi_n(v) \{ \text{Log sin } \frac{v-s}{2} - \text{Log sin } \frac{v-s'}{2} \Big| dv$$

$$+ \int_{s - \frac{s'}{2} + \frac{\pi}{4}}^{\frac{\pi}{4} + \frac{s'}{2}} \Phi_n(v) \text{ Log sin } \frac{v-s}{2} dv \Big| .$$

g) $$\Big| \int_{\frac{\pi}{4} + \frac{s'}{2}}^{\frac{\pi}{2}} \Phi_n(v) \{ \text{ Log sin } \frac{v-s}{2} - \text{Log } \frac{v-s'}{2} \} dv \Big| ,$$

$$\leq \int_{\frac{\pi}{4} + \frac{s'}{2}}^{\frac{\pi}{2}} |\Phi_n(v)| \times \frac{|s-s'|}{2} \times K \, dv < \epsilon ,$$

for $\epsilon > 0$ given providing $|s-s'| < \delta_5$ and using note one and the fact that $|\Phi_n(v)|$ is integrable in the interval of integration.

h) $$\left| \int_{s-\frac{s'}{2}+\frac{\pi}{4}}^{\frac{\pi}{4}+\frac{s'}{2}} \Phi_n(v) \text{ Log } \sin \frac{v-s}{2} \, dv \right| \leqq |s-s'| \, Q < \epsilon \quad ,$$

providing $|s-s'| < \delta_6$ for $\epsilon > 0$ given and using the boundedness of the integrand in the interval of integration.

Then for $\epsilon > 0$ there exists a $\delta > 0$ such that providing $|s-s'| < \delta$ then $|F_n(1,s') - F_n(1,s)| < \epsilon$ for $0 < s,s' < \pi/2$. The remaining cases follow in the same manner so that in the interval $[0,\pi]$ $F_n(1,s)$ is continuous. A similar but somewhat simpler proof is necessary to prove the imaginary part $G_n(1,s)$ is continuous in this interval.

APPENDIX II

Suppose we wish to evaluate the integral $\int_a^b f(x)dx$ and $f(x)$ cannot be integrated directly. Numerically this can be done by an approximation procedure known as the method of tangents. It consists of partitioning the interval $[a,b]$ into n subintervals $[x_{i-1},x_i]_{i=1}^n$ where $x_o = a$, $x_n = b$ and $\Delta x_i = x_i - x_{i-1}$ and then evaluating the expression

$$\sum_{i=1}^{n} f(x_{i-1} + \frac{1}{2}\Delta x_i)\Delta x_i \quad .$$

If $f(x)$ is integrable then this expression can be made to approximate the integral as closely as one wishes by letting $\underset{i}{\text{maximum}} \{\Delta x_i\}$ tend to zero.

APPENDIX III

Computation of $F_2(x,o)$

Example 1

x'	$F_2(x')$
100.	- .061
15.	- .037
5.	- .020
1.5	- .056
1.05	- .103
.95	+ .085
.05	+ .041

Example 2

x'	$F_2(x')$
1.05	- .0009
5.0	- .00045
15.0	- .00027

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